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ABSTRACT

"NCTM Student Math Notes" is published five times a year as a supplement to the "NCTM News Bulletin." This document consists of the five issues published during 1990. Each issue is devoted to a single math topic. The five topics represented here are: (1) "Concept of Number"; (2) "Basic Skills: Development of Number Concepts" (a continuation of "Concept of Number"); (3) "Mathematics and Writing: Showing All Work--Writing in the Mathematics Curriculum: An Introduction;" (4) "Math Self-Help Video Program" including student work sheets; and (5) "Dyslexia, Dyscalculia, and Some Remedial Perspectives for Mathematics Learning Problems." (CW)

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CONCEPT OF NUMBER

By MAHESH C. SHARMA

Number dominates our daily lives. Understanding number and its manifestations such as quantity, counting, and numerical operations is a fundamental human skill. Apart from language, calculation is perhaps the only culturally determined, semantic system that the majority of the population is expected to acquire and master. For success in today's highly technological world, it is important to have a good numbersense.

> "The bulk of the technological advancements made over the past three centuries, and particularly in this century, can be attributed to our species' quantitative skills" (Bjorkland, 1989).

In such a world, one needs to be literate and also numerate. The word numerate means the possession of two attributes:

> a. An 'at-homeness' with numbers, and an ability to make use of mathematical skills which enables an individual to cope with the practical mathematical demands of his everyday life.
>
> b. An appreciation and understanding of information which is presented in mathematical terms, for instance in graphs, charts or tables or by reference to percentage increase or decrease. (Cockcroft, 1982, p.11)

Man, even at the lowest levels of cognition, possesses the faculty which manifests itself in numbersense. Of course, its development is dependent on related experiences and its use. Even some animal and bird species seem to possess a recognition of quantity. They can recognize that something has changed in a small collection when, without their direct knowledge, an object has been removed from or added to a collection of few objects. This ability may be termed as rudimentary numbersense. There is no counting involved here. That is because when an object is removed from or added to a larger collection, there is no recognition of the loss or gain. Human beings also have similar innate sense of quantity. But the conscious act of counting, so far as we know, is an attribute exclusively human.

> "Thinking in numbers (and subsequently in terms of mathematical relations) is not innate but, like other aspects of cognition, has a developmental history" (Bjorkland, 1989).

It is a learned skill; facilitated by the demands and provisions of one's culture. There are few human populations who do not have some idea of number and its applications. Cohn (1968) in the study of developmental dyscalculia observed that:

> "It is rare to find a child who has been to school and yet has no quantity concept or no ability to symbolize these concepts, particularly if he has even a passable ability to 'write' or 'depict' (draw) his name ... It is also quite unusual to find a young child who cannot produce letters to represent his name but can nevertheless produce the first five number symbols."

The concept of number is deceptively a very simple process yet many children have difficulty in mastering and applying this skill according to a socially acceptable and agreeable timetable and pace. To most adults, the knowledge and use of the first nine natural numbers (one, two, three, ... up to nine) appears to be a very simple and straight forward process. Yet an average child in a normal learning situation takes around five years, from about the age of two to the age of seven, to learn to handle such numbers consistently and to apply them to a variety of everyday situations to solve simple quantitative problems. And the period becomes even longer if the use of arithmetic operations is included.

On first sight it appears that learning to count is merely a matter of reciting a string of words like a nursery rhyme, a feat that young children can master surprisingly early. The fact that basic number development proceeds so slowly for many children may seem surprising, especially in comparison with the relatively prodigious speed at which most young children appear to pick up language (Dickson, Brown, & Gibson, 1984).

> "The 18-month-old may respond with a loud and definitive "Two!" when asked how many cookies he

wants, and the 3-year-old may point to her dolls, saying "One, two, three, seven, nine, twelve-teen, threeteen, seventeen." Each is displaying some knowledge of numbers yet neither is using numbers with the same consistency and meaning that a 7- or 8-year-old child uses them. Primitive notions of number (or at least quantity) can be found very early in development: new-borns can discriminate between arrays differing in only one element (Antell & Keating, 1983). Yet a full understanding of the number concept is not obvious until the early school years, and possibly beyond" (Bjorkland, 1989).

Most children's first and best developed mathematical competence is counting. Almost every child acquires the concept of number and masters the skill of counting meaningfully sooner or later in his or her life. In an average child it is outwardly an uneventful development. Most chldren acquire the skill of counting with ease, but some with relative difficulty. Similarly, most children acquire the number manipulation skills (arithmetic operations) in the course of their normal development. Yet school mathematics is difficult for many children.

The ability to acquire and understand the basic number concepts relies on several prerequisite skills that develop, at some points, independently of each other and, at others, simultaneously. The process of learning number, then, is not a simple, step-by-step, linear one. A true understanding of number depends upon the interweaving of various factors with the appropriate timing. But, the learning disabled (LD) child may not be able to accomplish this interweaving of several preskills independently and without special teaching and experiences. For example, as we shall see, the art of counting involves pointing to one object at a time and keeping track of which objects have already been counted. These two processes are difficult for many young children to coordinate meaningfully and with consistency.

Teachers of young children, particularly those working with children with learning disabilities, need to know how children acquire numbersense and factors that are responsible for its optimal development. They should also know what factors are responsible for the difficulty. In this issue of the **Math Notebook** we plan to explore:

a) the ways children acquire the concept of number;

b) why some children have problems in learning numbers;

c) the nature of some of the problems experienced by children related to number; and

d) instructional suggestions for teachers in order to help children acquire the concept of number.

1. The Concept of Number: Assumptions and Approaches

Since the concept of number is such a fundamental part of human knowledge, many philosophers and epistimologists have commented on the concept and the processes involved in acquiring numbersense. Each one of them approaches the concept of number from his unique perspective. For example, the mathematicians Henry Poincaré and L. E. Brouwer have held the view that the number concept is a product of primitive intuition, preceding logical notions. Great mathematician and logician Sir Bertrand Russell, on the other hand, has supported the idea that number is a purely logical concept. Two opposing views.

Piaget believes that children acquire the concept of number by manipulating concrete objects and abstract the concept of number from these concrete experiences. He suggests that children must grasp the principle of *conservation of quantity* in order for them to develop the understanding of the concept of number. John Stuart Mill (1846) relies on perception as the source of understanding of number when he says, "that two and one are equal to three is a truth known to us by early and constant experiences: an inductive truth, and such truths are the foundations of the science of number." Accordingly, he considers learning number as an "evidence of sense." To him, it is a process of abstraction achieved by each child individually as a result of his interaction with his environment. In that sense, Piaget has elaborated Mill's ideas of number acquisition and has provided a helpful chronology of this development in a child.

Einstein believed and asserted that "the concept of number is a free creation of thought, a self-created tool which simplifies the organizing of certain sensory experiences. Logician Gottlieb Fregé (1950) said that "number is neither spatial and physical, like Mill's piles of pebbles, nor yet subjective, like ideas and impressions." John Dewey, in the same manner, warned that "number cannot be taught by the ... presentation of things, since number is not a property of the objects which can be realized through the mere use of senses ... It is a process of abstraction."

Luriya in his book **On the Pathology of Computational Operations** considers the development of number and arithmetical operations in children parallel to their development of facility in semantic structure akin to language. Number is a generalization of concrete experiences when he says:

> "Every number conceals a generalization; combinations of individual numbers arranged in a fully defined logical order can express any quantity."

These various ideas about number, propagated by different philosophers, although coming from diverse perspectives and approaches to the understanding of number, when seen in their totality and in realtion to each other suggest that the *development of the concept of number, in every child, begins with the help of sense perception, resulting into abstractions from these sensory experiences and then, the number, finally, is identified as a logical entity. The child organizes these, eventually into different collections supplying them attributes and properties. These collections of numbers, finally, possess logical structures, some of their properties akin to a language.*

All of this is a developmental process which is uniquely individual in rate. Yet, this development follows a remarkably common process in structure. The appearance of familiar milestones in the understanding of number, in that sense, also follows a common history.

Therefore, the abilities needed by the child, for the complete understanding of number, include an intact perceptual apparatus and a level of logical understanding that is capable of abstractions and making sense from the diversity of multi-sensory experiences. And, finally, mathematics learning, including the properties of number and its use, is closely tied in with the development of a well-formed formal language.

2. Meaning of Number

We encounter number in many forms. Understanding number means knowing how to use it. One use of number is to specify the size (numerosity) of a collection of objects. This is known as the *cardinal* aspect of number. The other use is assigning a number to denote the position of an object in a sequence (or in a collection). This is known as the *ordinal* aspect of the number.

In his book, **The Child's Conception of Number** (1952), Jean Piaget stated that the concept of number has three basic aspects: *cardinal number, ordinal number*, and *unit*. He has given criteria for determining when a child understands each of these basic concepts. Piaget concludes that in order to really understand, to have a complete grasp of even small natural numbers and to apply natural numbers, an appreciation of both ordinal and cardinal aspects is needed. It involves forming relationships between the ordinal and cardinal aspects of numbers. He believes that this is acquired at the same time that many other logical operations develop, at around 6 to 8 years for average children (Dickson, Brown, & Gibson, 1984).

This often means the child's facility to manipulate the numbers in flexible ways: for example, (a) to be able to count forwards or backwards and (b) recognizing clusters of objects without the total dependence on the ordering of objects involved in the collection. It seems that the lack of this flexibility is responsible for the delay in acquiring complete understanding and consistent use of number by children. To add to this complexity are the various meanings of number warranted by their use in different contexts. The meaning of number is dictated by its use and vice-a-versa.

a. Series Meaning of Number

In this form, understanding the meaning of numbers from one to ten may mean the knowledge that 'one' means a single thing of the sort named, that 'two' means one more than one, that 'three' means one more than two, and so on. This is known as the *series* meaning of number. To know the meaning of 'six' in this sense is to know that it is one more than five and one less than seven--that it is between five and seven in the number series. Use of number line and other counting materials such as beads, counting blocks, etc. are effective means of understanding this form of the meaning of number.

In this form of number, the emphasis is on the sequential aspect and meaning of number. This becomes the underlying assumption in developing most commonly used teaching techniques to introduce the idea of number to children. Such techniques are better suited to children with quantitative mathematical learning personality (Sharma, 1979; 1989). (Children who have quantitative mathematics learning personality rely heavily on procedures and seqential aspects of mathematics).

b. Class or Collection Meaning of Number

The knowledge that two represents a collection of two units, and that three fits a collection of three units, each number being a name for a certain sized collection of discrete objects, such as apples, pennies, boys, balls, fingers is known as the *collection* or *class* meaning of number. In this sense, to know the meaning of six is to be able to name correctly any collection of six separate, easily distinguishable individual objects. The use of different types of collections of objects such as counting blocks, Unifix cubes, etc. are good means of developing this form of the meaning of number.

Here again the sequential and discrete nature of number is being emphasized. Once again children with a quantitative mathematics learning personality learn the number better in this form and even prefer this approach to number (Sharma, 1979; 1989). Most experiences with number and formal teaching strategies in a school setting begin with this aspect of the meaning of number. Children are shown collections of objects and then they have to identify and recognize a collection by counting and relating the collection to a specific number and vice-a-versa. In this process the child understands the number as the number assigned to the last objects in the counting process such as when the child says: one, two, ..., seven. For him seven represents the count of that number. This type of teaching strategy has the disadvantage that many children do not associate the one-to-one correspondence between the numbers and the objects in the collection. For example, since the child utters the word seven in association with the last object he concludes that there are seven objects in this collection, but he may also associate seven with the property or the name of the last object. Therefore to some children, the number becomes the identity and property of the last object counted rather than the property of the collection. This situation, generally, persists until the child understands the concept of one-to-one correspondence well. Some children may recognize the number of objects in the collection by subitizing (visual observation). In this case the emphasis is not on the sequential counting aspect of the collection.

c. Ratio Meaning of Number

Young children develop 'relative codes' before 'absolute codes' i.e. they learn to distinguish which of two shapes is larger or smaller, which of two lines is longer or shorter, and so on, before they learn to judge absolute size, length, direction or numerosity (Bryant, 1974). Therefore, in yet another sense, the knowledge of the number from one to ten may mean knowledge that two is twice whatever is called

4

one, that three is three times whatever is called one, and so on. This is, of course, the *ratio* meaning of numbers. To know the meaning of six in this sense is to know that if a "—" is one, then a line drawn by putting six of these units together to form "______________" represents the number six.

The use of materials such as Cuisenaire rods, where the length or color of each rod is an identifier of a particular number, helps develop this form of the meaning of number. The understanding of this meaning of number is the precursor to understanding the concept of measurement. We have observed that qualitatively orientated children (Sharma, 1979; 1989) prefer this aspect of number meaning. (Children with qualitative mathematics learning personality are those who perceive whole-to-part relationships and recognize patterns and relationships first rather than the sequential aspects of the information.)

d. Relational Meaning of Number

Another way of understanding number is by comparison: a number may be a smaller or larger fraction of another. This means understanding relationships between numbers. For example, to know 'six' in this sense is to know that it is more than five or four; less than seven or eight; that it is two groups of three put together or it is three groups of two put together; that it is the sum of five and one, or of four and two, or of three and three; that it is two less than eight; that with four it makes ten; that it is half of twelve, and the like. This is the familiar 'nucleus of facts' or *relational* meaning of number.

The family of facts idea is related to this meaning of number and this meaning of number is the beginning of the computational aspects of numbers (arithmetic facts). Once again, materials such as Cuisenaire rods, Dienes' and multibase blocks are very good means of developing this meaning of number. Acquiring an understanding of this form of number indicates that the child knows not only the meaning of individual numbers but also the relations between them.

e. Implications of the Various Meanings of Number

This information has implications to teaching, both for the methodology and choice of instructional materials, not only the number concepts but all of arithmetic. Number is conceived, understood, and used in various forms. Some things are to be counted and others are to be measured, and in some other situations the number of objects in collections are to be perceived in just one gestalt. One meaning of number or one approach to teach it is not enough for its optimal conceptualization. It should, therefore, be obvious that the various meanings of number have claims upon the attention of the child. To know the number means to know it in all of these respects. The difficulty arises when curriculum planners and teachers take a narrow view of the meaning of number.

Different meanings of the number appeal to children in varying degrees, we should still help children develop various meanings of number. Some children, particularly those who have quantitative mathematics learning personality, prefer the series and nucleus of facts meanings of number and they use counting strategies to master the concept of number and the number facts. The instructional approaches, in which the number is introduced as a representation of discrete quantity seem to work better for them. Therefore, initial teaching activities may begin with these kinds of materials. On the other hand, the children who exhibit characteristics of qualitative mathematics learning personality seem to prefer the collection and ratio meaning of the number. They seem to acquire number concept easier with the help of those instructional approaches where characteristics such as the size (i.e., length, area, etc.), color, relationships and the like are emphasized. Since, in every classroom, there are children with various mathematics learning personalities, it is important to use both quantitative and qualitative strategies and materials to teach number and other mathematics concepts.

In ordinary school practice, many teachers emphasize only one meaning of number and present the number concepts in that manner. This means that the children's conceptualization of number is limited to that one meaning, when the experience with all meanings is essential for complete understanding of number.

Same limitation in meaning of number is also evident in the textbook presentations. Most textbooks present a single strategy for teaching the concept of number. In most cases it is the discrete strategy. Most teachers because of their dependence on text-books, use the "counting up" or "adding on" strategies to derive the number facts. These presentations lead the child to believe that number is the outcome of sequential counting only. They perceive number in its discrete form and use discrete counting strategy to derive number facts and many of them transfer the discrete counting to counting on fingers. It is a good introductory strategy, but this does not create a good understanding and appreciation for number. It is an inefficient strategy. It is important for a child to replace these introductory inefficient strategies. Most children are able to transcend these limiting strategies and may even invent efficient strategies of their own, but some average and most LD Children find it difficult to replace these strategies. They need help and more experiences in order to be able to do so. Exposure to different meanings of number through diverse materials and multiple teaching strategies will help them replace their inefficient strategies with efficient ones.

Every child needs the integration of the various approaches to the different meanings of the number.

3. Young Children's Number Understanding

Before examining, thoroughly, the intellectual processes needed for developing numbersense and counting in children it is important to consider the following findings and propositions relating to children's understanding of number:

(a) Before entering school, and outside the context of formal education, children develop various techniques for solving quantitative problems. For example, by the age of 4 years, children can easily see, without counting, which of two sets, randomly arranged, has "more" objects (see figure 1) than the other,

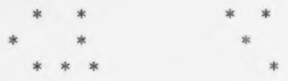

Figure 1. Two randomly arrayed sets.

This is true at least when a relatively small number of objects are involved in the sets under consideration. Children solve this type of task by visually comparing the relative areas occupied by each set. This is possible because of children's visual/spatial capability. This technique is reasonably effective because area is correlated (imperfectly) with numerosity, particularly when one of the collections has less than five objects.

On the surface, this observation, may give the idea that children understand the concept of more or less. That may not be the case. A young child of age two or three, for example, learns to distinguish between two toy cars and three toy cars merely by looking at the group of cars, in just the same way as he learns to use his visual perception to distinguish between a car and a bus, or between a red car and a green car. This process is based on perceptual discrimination, but this has important pedagogical implications. In most teaching strategies relating to number, a child is introduced to number through sequential counting only. This is a limiting strategy. We must introduce children to other strategies too. For example, it is essential to capitalize on the visual/spatial capability of children in the development of number. But very little effort is made to do so.

(b) Historical and anthropoligical research relating to the development of number concepts shows that counting-based techniques, usually involving counting on fingers and other parts of the body are spontaneously developed by children as a result of their interaction with the environment and can be extraordinarily successful for a variety of rudimentary arithmetic operations involving simple practical problems. Although the counting strategy, ultimately, is not an efficient one, but, this clearly shows that formal education is not a prerequisite for rudimentary practical arithmetic. Individuals in many societies (where formal education is not available to many of its members) have very good number conceptualization and highly effective arithmetic calculation techniques and yet may have no formal education or experiences with written arithmetic. Therefore, number conceptualization by children is not purely a function of formal teaching. Some societies have counting and arithmetic skills embedded in their cultural activities and children in these societies develop numbersense. Once again, teachers should use these naturally developed counting and ethnomathematics skills to develop number conceptualization including formal counting and other meanings of number.

(c) There are discontinuities among different areas of children's work related to number. The most common type involves a gap between the child's written arithmetic and his or her informal, counting based procedures or invented strategies. Many children can add on their fingers but not on paper. For most young children, counting or mental calculation is often more effective than written work. Indeed, within the area of arithmetic, counting seems to be the most basic skill (Ginsberg 1975a,b). Therefore problem formulation and problem solving using number should begin using informal techniques and be followed by symbolic representation of these experiences by recording the outcome.

Informal mathematics of both types, that is, both with and without counting, represent generally unsuspected cognitive strengths in children who otherwise have difficulties with arithmetic and mathematics. On entering school, children possess an imposing array of spontaneously developed and powerful intellectual processes on which arithmetic and mathematics learning can build. The heavy emphasis on written arithmetic, too early in the schooling process, may sometimes put a halt to the child's spontaneous development of number concepts. Teachers must preserve the continuity of children's natural development of ideas relating to number during early school experience.

4. Foundations of Number

There are important landmarks in fully understanding the concept of number which most children seem to arrive at between the ages of 2 and 8. Research on number acquisition (Descoeurdes, 1921; Piaget, 1952, 1968; Dodwell, 1962; Greco & Morf, 1962; Elkind, 1964; Beckwith & Restle, 1966; Bever, Mehler, & Epstein, 1968; Potter & Levy, 1968; Biggs, 1970, 1985; Wang, Resnick, & Boozer, 1971; Gelman, 1972; Freudenthal, 1973; Bryant, 1974; Copeland, 1974; Schaeffer, Eggelston, & Scott, 1974; Brainerd, 1977; Ginsberg, 1977; Gelman & Gallistel, 1978; Siegel, 1978; Donaldson, 1978; Donaldson & Wales, 1978; Fuson, 1980) shows that the number acquisition is a complex yet a developmental process. Most researchers realize that in one form or another the concept of number is organized in close connection with the gradual elaboration of systems such as *classification, counting, pattern recognition, inclusions* (ascending and descending hierarchy of *logical classes*), systems of *asymmetrical relations* (qualitative and quantitative seriations and ordering), and the development of concepts such as *one-to-one correspondence* and *conservation of number*. For example, the critical components of the concept of number have been defined by Piaget and Inhelder (1969) as classification (specifically class inclusion), seriation, one-to-one correspondence and conservation of number. Therefore the understanding of number results from an operational synthesis of groupings (classification), ordering (seriation), counting, pattern recognition, conservation of number and one-to-one correspondence.

These concepts and the activities that help develop them in the initial stages are heavily concrete and are facilitated or hindered by sensory actions and perceptions. Number conceptualization in this sense is dependent on sense perceptions, actions on concrete objects and the resulting abstractions from them.

There are definite stages most children go through in acquiring the concept of number. This includes number as a representation of the counting activity (cardinality of a set), number facts and also in expressing definite relationships between numbers. The number and its operations, according to Piaget (1952) "lead from intuitive and egocentric pre-logic to rational coordination that is both deductive and inductive." He further contends "that the construction of number goes hand-in-hand with the development of logic, and that a pre-numerical period corresponds to the pre-logical level." According to Piaget, this development of logical thinking is intricately related to children's operational thinking. Quite young children, according to Piaget's research, do not think operationally at all. They can act upon something in the environment, but once an action has been performed, they are unable to keep in mind the way things looked before. Thus, they cannot mentally reverse their actions; in the technical language of Piaget, they have not achieved *reversibility of operation*. According to Piaget, children are heavily influenced by the sensory and perceptual nature of the events that surround them during this early stage of intellectual development. The way things are presented to them are accepted as the way things are. Mental transformations, so characteristic of older children's and adult's thinking, cannot be performed. Perceptual features dominate thinking to a much greater degree (Resnick, 1981).

Dependence on perceptual features of objects or arrays and the inability to think reversibly are characteristic of what Piaget has labeled preoperational thinking.

Extensive work by Piaget (1952, 1968) and Schaeffer et al (1974) with children on their preparation for and understanding of number concepts provides some guidelines in determining what is involved in understanding children's ability to acquire and master the concept of number. For example, Schaeffer, Eggleston and Scott (1974) have identified three stages for children's development of number concepts. These are very similar to the Piagetian stages of number conservation.

Stage One; 'Not able to count correctly collections of five or more objects.' The Stage One children are characterized by the following:

a. Recognize the number of objects.in very small collections, certainly one and two and sometimes three or four, without counting them (probably by recognizing a visual or auditory pattern although sometimes possibly by counting); They can discriminate between small numbers on a basis of the perceptual pattern. Descoeudres called this the 'one, two, three, a lot' syndrome.

b. Distinguish between larger and smaller collections in cases where at least one collection was less than five, both visually and verbally; Children develop the idea of 'more' or 'less' in this stage (Schaeffer, Eggleston, & Scott, 1974; Donaldson & Wales, 1970; Donaldson & Balfour, 1968; Siegel, 1978). However, the ability to judge which of two collections is larger clearly depends on how the collections are arranged. When two equal collections were arranged in rows of different lengths children at age three and a half years, and beyond, tended to judge the longer one as the more numerous, even when there were only three or four objects in each set (Piaget, 1968; Bever, Mehler, & Epstein, 1968; Gelman, 1972).

c. Distinguish between larger and smaller collections of any size provided they were lined up so as to show one-to-one correspondence or lack of it.

Most children, during this stage, have difficulty counting five or more objects. According to Schaeffer's view Stage One children have grasped the cardinal aspect of number i.e. using number to determine the size of a collection of objects, at least for very small collections, but do not yet have the ordinal aspect associated with attaching a sequence of numbers named to a sequence of objects.

Stage Two: 'Children could count at least one array of five or more objects correctly, but correctly applied the 'cardinality rule' less than half the time.' (The cardinality rule is the knowledge that the number on which the counting process finishes can be used to denote the size of the set.). The Stage Two children are characterized by the following:

a. **Pattern recognition:** These children can recognize small numbers as patterns but they are more likely to count them than are Stage One or Stage Three children. Although Schaeffer et. al. do not give reasons for this shift from pattern recognition to counting, we believe that formal teaching of counting and teachers' emphasis on sequential recognition of objects might be responsible for this reliance on counting. Parents also emphasize sequential counting at home, which also contributes to their reliance on sequential counting. The pattern recognition appears back once again when children have familiarized themselves with numbers by repeated experiences with counting. Thus Stage Two children have acquired the concept of counting, i.e., "repetition of a string of number names in an identical order--the stable order principle" and "each number name must be matched to one and only one object--the one-to-one principle." However, these children have not yet acquired the cardinality aspect of number (the inability to isolate the size as a stable characteristic of the collection) for collections larger than five. They do not connect the counting process with its result, which is that the final number represents the total size of the collection, and this is invariant i.e., it does not depend on the order in which the objects are counted. They have grasped the ordinal aspect of number (i.e., assigning numbers to a sequence of objects in a counting process) and understand the cardinal aspect for very small collections, but they have not as yet related the two aspects for numbers greater than five.

Stage Three: The criterion for Stage Three is that the child can apply the cardinality rule i.e. he can use the counting process to arrive at the size of a collection, and consider the size itself to be a stable property of the collection. However, Stage Three children unlike those at Stage Four, do not recognize that, for instance, seven is necessarily more than six.

a. **Pattern recognition:** The Stage Three children are more likely than the Stage Two children to recognize the number by the pattern of objects in a small collection (containing 1 to 4 objects) without counting them. Gelman also reports that children are increasingly likely to identify the total size of the collection without counting as they get older (at least in the 3 to 5 year old age group), even when they are explicitly asked to count.

b. **Counting:** Although the Stage Two children seemed to have grasped the basic essentials of the counting process, the Stage Three children were more accurate, presumably because the process itself had become more automatic. They have begun to connect the ordinal aspect of numbers used in assigning numbers in sequence to a series of objects with the cardinal aspect of assigning a number to represent the size of a collection.

c. These children have made the connection between the process of counting and its application in giving the number of objects in the whole collection.

Stage Four: The criterion for Stage Four was the ability to recognize the larger of any pair of numbers up to ten.

The characteristics of Stage Four children were very similar to the Stage Three children, except that their counting of numbers up to ten was accurate in each situation at least 98 percent of the time, and they were even more likely to recognize by sight numbers up to four. They also chose the larger of two numbers when asked to choose the larger of the two sets. Thus they have firm grasp of counting and its application to distinguishing between the relative sizes of two collections, at least when two collections contained ten objects or less. This stage is reached by most children between the ages of five and six.

5. Language and Number

Although, linguists suggest that language development precedes cognitive abilities (Chomsky 1968), the cognitive psychologists suggest that cognition comes first. For example young children acquire number concepts earlier than they do pluralization rules. The Piagetian school (Piaget & Inhelder, 1969) takes the position that language does not constitute the source of logic but is, on the contrary, structured by it. Piagetian group concludes that cognitive development proceeds on its own and is followed by, and reflected in language.

Beilin and Kagan (1969) have shown that to a large extent the language development does not effect the acquisition of number concepts. They concluded that the development of linguistic rules does seem to be associated to a limited extent with the speed of concept attainment but not with concept reversal. Their findings suggest, "that the correlation may be due to the intervention of language in this type of problem solving and conceptualization tasks."

We believe that language alone (without any or very limited concrete manipulative experiences) cannot effectively develop the concept of number in young children. Similarly, concrete experiences alone are not enough for the abstraction of number from specific concrete activities to general notion of number. Every concrete experience needs an appropriate and stable label and the language provides such labeling and abstractions inherent in them. Therefore, number acquisition is facilitated by language if appropriate concrete experiences are present. Concrete experience is the starting point for number conceptualization and language accelerates the symbolization process and helps in abstracting number from these concrete experiences. Number conceptualization is not an artifact of language. Language plays a role, but it is a limited one. Mathematics is a language in itself and the numbers are its alphabet, symbols, and its smallest thinking units. Mathematical symbols, concepts and its language modify the natural language and it in turn helps shape mathematical thinking, that is, if mathematical thinking has taken hold.

In the *natural language* system, numbers function as predicators, modifying the nouns in their scope and meaning quantitatively, such as five dishes, two toys. As a result, they function within a sentence like adjectives in some respects.

In the *formal language* of mathematics, on the other hand, numbers do not, generally, function as predicators, on the contrary, number characteristics are described by using other predicators. For example, six hundred billion is a very big number, or is an even number, or is much smaller than the number six hundred million billion. The question "seventeen what?" is generally irrelevant and meaningless in the context of the mathematical language. Thus, in the formal language of mathematics, the numbers are abstract singular objects (Fregé 1884/1968; Geach & Black 1952; Nesher 1972; Quive 1960; Russell 1971), but, at the same time, to mathematicians numbers are very concrete entities.

Acquisition of number concept, then, requires that the child be able to perform a particular type of abstraction, which in one sense represents the transition from the conception of number as a predicate in natural language to its conception as an object in the language of mathematics (Nesher & Katrid 1986). For some children, particularly LD children, this transition is not an easy task.

The transition from natural language to a formal mathematical language system and abstractions from sensory concrete experiences to formal entities are important processes necessary for successful understanding of mathematics, particularly the number concepts. It is an ontological process. It evolves through a prolonged constructive process which begins with natural language and concrete experiences and ends in the child's operational conception of number as part of the formal mathematical language system. For example, according to Nesher & Katrid (1986) "from a language standpoint the interrelationship between the predicative function of numbers in natural language and their status as objects in the language of mathematics is a constant accompaniment to the process of arithmetic learning and instruction in the primary grades."

The mediation between the natural language and the formal language of mathematics is an important process which is dependent on a variety of concrete and linguistic experiences. It can be seen as a three stage process.

a. At the **first stage** of the process, the children's familiarity with numbers is based on their encounters with numbers in natural language and it is purely in concrete form. In this stage, they seek out the objects to which the numerical expressions apply.

b. At the **second stage,** mathematical learning relating to number consists of the ability to make a transition from the natural language to formal language, but lapses are possible.

c The **third stage** appears when children have acquired the ability to use the formal language of mathematics. At this stage the child can handle numbers as abstract objects and must make the transition in the opposite direction. That is, must apply the mathematics models to appropriate concrete situations.

6. Mastering the Concept of Number: Rote vs Understanding the Meaning

As we have seen, children begin counting by rote, at least small numbers, much earlier than they come to school. This counting in many cases does not mean real understanding of number or its effective use. The concept of number or cardinal property evolves in the normal course of development around the age of 6.

Counting, as a rote activity, can be taught quite easily, but the student may or may not know what he is doing except saying the words in a sequence. Counting behavior--the recitation (verbal or tactile) of an ordered sequence of responses--to be useful must be supported by the underlying cognitive structures and processes. There are many young children of ages 4 through 6, who can count objects in a rote manner where the last number uttered represents the cardinality of the set they are counting, but they may not know what that means. This was made clear when I was working with a seven-year child in a tutorial setting:

Tutor: Can you tell me how many pennies there are? (Points to a collection of pennies on the table.)

Child: (Child counts by touching each penny once) Seven.

T: When you were counting these pennies what number came just before seven?

C: I do not know. Is it one? Three? Five? I don't know.

T: Please count the pennies again. (Points to the same collection of pennies again)

C: (The child counts the pennies again using the same strategy of touching each one as he counts them.) Seven.

T: Can you give me six pennies?

C: I don't know. I don't think I have enough here. Maybe I do. (He counts six pennies one by one.)

This example shows that the child knows the number words but does not know what those number words mean. He has learnt these number words by rote and without understanding.

The distinction between two ways of learning mathematical concepts: senseless drills vs understanding relations has a long history in the psychology of mathematics learning. **According to Wartheimer (1959; 1967), although both** methods, learning by rote and learning by meaning, provide the same level of performance on an immediate mastery test, the understanding group forms a broader learning outcome capable of superior transfer and long term retention. This distinction applies to all concepts in mathematics including number conceptualization.

Scandura (1970) has offered a similar distinction and further suggests that "what is learned" can be viewed as a rule system. The same mathematical skill can be supported either by a set of simple nonrelated rote activities or comprehensively understood associations between different components of a concept. For example, Ehrenpreis and Scandura (1974) taught a mathematical skill by providing either a set of discrete low-level rules or a system of higher order rules which could be used to generate the lower rules. The two instructional groups performed similarly on tests of specific transfer but the higher order group showed a clear superiority on general transfer tasks.

Mayer (1977) summarizes several studies on systems of learning mathematical concepts and says,

> "the present experiment explores the idea that counting behavior can be learned by acquiring different types of underlying cognitive structures and that certain instructional variables can influence which type of structure is acquired. In particular, meaningful context that is available during learning may result in a broader learning outcome than presenting the same context after the subject has learned to give the correct chain of responses."

In that sense, the understanding of the different meanings of number is difficult to acquire without the manipulation of concrete materials of diverse kinds and may be dependent on the type of learning/instructional materials used. Therefore, the instructional materials to be used by teachers should range from counting blocks, dominoes, number line to Cuisenaire rods, playing cards to visual cluster cards (see **Math Notebook** Volume **2**, Number 10, 1981 on visual clustering).

Note: Continued in the next issue.

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BASIC SKILLS: DEVELOPMENT OF NUMBER CONCEPTS II

(MAHESH C. SHARMA)

(Continuation from Volume 8, Numbers 1 & 2)

Gelman and her colleagues (Gelman & Gallistel, 1978; Gelman & Meck, 1983; Gelman, Meck, & Merkin, 1986) found that preschool children generally, have a firm grasp of the principles which underlie counting. Many children, by the age of five, know that

(a) number words should be produced in a fixed order (stable order),

(b) objects can be counted in any order (order irrelevance) so long as every object is counted once and once only (one-to-one principle), and

(c) the last number in a count represents the number of items in the set (cardinality).

This information is important to know about children in order to plan school experiences for them relating to number but the findings are true only as long as the number of objects in the set is not more than four. And, in many cases, this mastery (counting the number of objects) is not transferable to solving simple problems involving number. Many children at age five and six, and some older children as well, make some glaring mistakes in tasks which can be solved by simple counting, or in problems that demand very simple addition or subtraction skills. This shows that although children are able to count sequentially they may not have complete understanding of number. It seems that the counting facility of this type and at this level is more of a rote nature.

For example, however good they are at counting single sets, they seem to have a difficult time as soon as they have to make comparisons between two sets. This shows the lack of one-to-one correspondence. Michie (1984) and Saxe (1979) have demonstrated how reluctant children are to count in order to make comparisons, and Cowan (1987) has shown that they avoid using one-to-one correspondence cues to compare two rows of dots even when they are given striking perceptual reminders of this correspondence. Therefore, to use number meaningfully, it is not enough to have a rote counting facility. The application of number mastery by children requires the true understanding of number.

In order to understand number truly and to be able to apply this understanding meaningfully, it is important that the child has a sound mastery of the underlying prerequisite skills.

As we have seen (see previous issue of this publication, Volume 8, Numbers 1 & 2), number is not something impressed on the child's mind by external forces, or just the perception of objects, but a product of the mind's actions on objects in the valuation of quantity. It is an interactive process, a system of internally symbolized representations. To use number meaningfully implies mental operations on the symbolized forms but, before the ultimate level of realization of number is reached, a child must proceed through many levels of relational understanding (i.e., the child is able to see what he is doing and why he is doing it) in which he employs logic to rationalize and deduce (Choat 1979, 1989; Skemp 1976, 1979, 1971). The relational understanding of number can come only when the child has a sound grounding in the prerequisite skills for number learning. Educational programs, both formal and informal, for children between the ages of four through six, should emphasize extensive work with manipulation of objects for the development of these preskills.

7. Prerequisite Skills for Number Conceptualization

There are definite prerequisite skills and support experiences that help a child conceptualize number more easily. Some of these important prerequisite skills are:

a. **Grouping/Classification:** Piaget considers grouping as the principle from which stem classification, seriation, conservation, and number understanding. During children's early experiences with their environment, they begin to identify objects and actions. Children begin to recognize objects according to their physical characteristics based on the sameness and differences of objects. This identification or recognition of objects of the same type or having the same characteristics is **classification.** The ability to classify grows out of the child's experiences with noting and acting upon resemblances and differences. In the initial stage this is guided by the physical characteristics such as: color, shape, or size of the objects being classified. The process by which children arrange the objects in some manner is called

ordering. Understanding logical classification (combining on the basis of sameness) and **seriation** (ordering on the basis of differences) are essential for number conceptualization. What is critical in any classification process is recognition of the specific qualities that characterize members in a group. Piaget points to groupings as being essentially of three kinds:

(i) Those that pertain to **identity** or **equivalence,** e.g., when a child says "my toy is the same as yours." In this process pairs of objects are related, and at this stage children follow no overall plan in grouping the set of items. They make decisions on an item-by-item basis. According to Inhelder and Piaget, these kinds of collections typify the classifications of children 5 years of age or younger.

(ii) Those that refer to the logical system of classes (**class inclusion**) wherein two classes may be included in the other, or may partially overlap, or may be mutually exclusive, e.g., all robins and sparrows are birds; birds include robins, sparrows, ducks, and other species; the class of birds also contain birds that swim; and finally

(iii) Those groupings that refer to **relationships** between the parts and whole of a concrete object or a collection of objects or persons. For example, this box contains wooden beads, some beads are black and some are white. When a child can answer a question such as: Are there more wooden beads or black beads?

By examining the composition of and justification for children's groupings, one can infer how children represent sets of items and the type of relationships they use for categorizing diverse stimuli. Thus, the preparation of a child for number experiences should include an extensive amount of work in grouping of objects and classification activities.

For example, young children eventually learn which animals are included under the rubric "doggies" and, a little later, which are excluded from the group. In the classes of number, what defines membership is quantity and none of the other possibly confounding features of size, configuration, or amount of space occupied. The confounding features create problems for learning disabled (LD) children. A learning disabled youngster could enumerate more distractors, therefore, missing the main identifying characteristics. Thus, the direct precursor of number is the ability to focus on and to discern quantity even in the presence of distractors (Cherkes-Julkowski, 1985, p. 82). This is one of the skills that clearly distinguishes the performance on counting tasks of LD children from normal children.

These early operations of classification and ordering are basic to the development of the concept of number. The child's classifying behavior progresses from simple to complex. The classification activities range from using single, simple physical characteristics (approximately 4 through 7 years); from observable physical attributes to abstractly defined terms; and from objects in the environment to concepts that are created in the imagination for a specific purpose.

The child who puts his toys together and a scientist who identifies a virus according to some given criterion are involved in similar activities, although at a much different level of sophistication and complexity. The child is using simple categorization as classification scheme, whereas the scientist is using abstract criteria for classification. One has to go through several stages of classification using abstract notions. Classification, therefore, is a never ending developmental process.

(i) Simple Classification (One Property)

Children begin to exhibit classification by one property early in life, beginning around age four. The first classification experiences for school and preschool children involve sorting of things that they encounter in their daily experiences. The criterion used at this level is that of "belonging". This doll "belongs to" me. Similarly, books "belong on" the shelf, pencils "belong in" the pencil holder. The basic structure used in classification at this level is perceptual and egocentric. It is based purely on a child's own experiences. Criteria suggested by others are difficult for the child to integrate with his own classification schemes. Only in the later stages of cognition that children are able to do this.

Although some children begin to classify objects by shape, most four year olds use color as a first criterion for classification and then they switch to shape, etc. Four-year-olds generally do not have an overall plan for classification at the outset of the activity — it is purely arbitrary and is conducted by looking at each individual piece in the collection. This classification process is not exhaustive as sometimes they may even miss some of the pieces to be classified.

(ii) Multiple Classification (Two Properties)

The idea of classifying objects using two or more characteristics simultaneously is called **multiple classification.**

Around the age of five or six, children begin to classify objects using two properties, such as: color and shape, shape and size, color and size, color and thickness, etc. The mastery of two property classification is not achieved by many children until the age of 7 or 8. Only when the child has basically mastered the concept, which Piaget calls "centeredness" (a characteristic of the preoperational stage of cognitive development) that the child is able to focus on the multi-property criterion. This mastery indicates that they are able to deal with the concept of union and intersection of sets which prepares children for the quantitative concepts of simple addition and subtraction operations.

(iii) Multiple Classification (Three Properties or more)

Around the age of eight or nine, children begin to classify objects using three or more properties simultaneously such as: color, shape, and size; shape, thickness and size; color, newness, and thickness; etc. The mastery of classification of three or more properties is not achieved by many children until the age of 9 or 10.

(iv) Subclassification and Hierarchical Classification

When children begin to classify objects using multiple criteria, they begin to deal with **subclassification** — all sea birds are part of the collection of all birds. The addition of the criterion sea (sea birds) gives rise to a subcollection. The mastery of the concept of subclassification is necessary for conceptualizing the concept of subtraction and the related ideas of division and fractions.

There are two types of subclassifications: **ascending** and **descending.** For example, when we represent a city on the map, the county containing that city, the state containing that county, the nation containing that state, and finally the continent containing the country, we are engaged in the development of the ascending hierarchy of classification. Whereas, if we reverse the order of subclassification it becomes an example of descending hierarchy of classification. The concept of ascending hierarchy begins at about 6 to 7 years of age and by the age of 9 most children have mastered the concept. The concept of descending hierarchy, on the other hand, is mastered by most children around age 11. Thus, the concept of descending hierarchy is considerably more difficult than ascending hierarchy. The concept of hierarchy and multiple classification are mastered almost at the same time by most children.

Apart from this scheme of classification, i.e., using criterion for classification of objects, children's classification processes can also be described from another perspective. Bjorkland (1989, p. 114) has given very good examples to illustrate these features of classification scheme, as outlined below:

(i) Ideosyncratic. Children between the ages of two to three generally group items, usually in pairs, using ideosyncratic justifications which may include their likes and dislikes. Very little or no attention is paid to any apparent physical or semantic properties of the objects to be classified. At this level, children either fail to provide a reason for their groupings or provide one that is independent of any physical or conceptual characteristics of the stimuli (Denny & Acito, 1974; Inhelder & Piaget, 1964; Vygotsky, 1962). Examples (Bjorkland, 1989):

★ "Horse and pants go together because that's a horse and those are pants."

★ "Cat and chair go together because a cat meows and you sit in a chair."

★ "Shovel and dog go together because I like shovels and I liked dogs."

(ii) Perceptual. Perceptual justifications are found mainly in the groupings of 3- and 4-year-old children but are also found in the classifications of older children. Items are grouped together on the basis of common perceptual characteristics. Examples (Bjorkland, 1989):

★ "Pie and hat go together because they're both round."

★ "Cat and squirrel go together because they both have long tails."

★ "Rabbit and milk go together because they're both white."

★ "Train and ship go together because they're both big."

(iii) Complimentary. Items are grouped on the basis of real or potential interactions as in real-world scenes or stories. Complimentary justifications increase in incidence over the preschool years and typically are replaced by conceptual justifications sometime between the ages of 6 and 9 years. However, complimentary justifications persist into adulthood and are not totally replaced by conceptual classification. Examples (Bjorkland, 1989):

★ "Cowboy, horse, and pants go together because a cowboy needs pants to ride a horse."

★ "Cat and milk go together because a cat drinks milk."

★ "Sailor, hat, and ship go together because a sailor wears a hat and works on a ship."

★ "Hamburger, knife, napkin, and menu go together because when you're at a restaurant, you get a menu and order a hamburger and use a knife and napkin when you eat."

(iv) Conceptual. Items are grouped on the basis of some common characteristic or inherent similarity such as being a member of a category or having similar functions. Conceptual justifications can be found in the classifications of preschool children, but they increase in frequency between the ages of 6 and 10 years. The use of conceptual classification seems to be related to formal schooling. Examples (Bjorkland, 1989):

★ "Coat and shirt go together because you wear them."

★ "Shovel, hammer, saw, and screwdriver go together because they're all tools."

★ "The man, woman, boy, and baby go together because they all belong to a family."

★ "Baseball, jacks, and tag go together because they're all games."

Children continue to use a variety of strategies over a broad range in order to classify objects. This multi-criteria approach indicates that the child is functioning at more than one level of classification. For example, a preference for conceptual classification is usually observed in children at 8 or 9 years of age, but complimentary classifications do not disappear altogether. Complimentary schemes continue to be used by older children and even adults when stimuli and task demands call for them (Greenfield & Scott, 1986; Rabinowitz & Mandler, 1983). When a child keeps on adding criterion in the process of classification, he is developing more and more subclasses and moving towards conceptual classification (Bjorkland, 1989, p. 116). Providing concrete experiences with diverse materials, then asking hypothetical questions about the classification criteria children use, facilitates an increase in the number of strategies they employ. When asked hypothetical questions children begin to clarify and elaborate on their classification schemes. For example, we have observed that children are able to retain more from concrete experiences when the concrete activities are followed by these hypothetical questions.

b. **Seriation:** The second component in understanding the number is seriation. Seriation refers to a child's ability to arrange elements according to increasing or decreasing size, graded heights, shades of colors, etc. In other words discerning and understanding the logical sequence that exists between things or events: the recognition of gradations of amounts and sizes. A child, for example, orders a collection of Cuisenaire rods from the smallest to the largest or largest to smallest. Initially the arrangement may happen either by trial & error or by chance. Slowly he begins to coordinate the difficult thought that there is a definite sequential relationship and notes the gradation. Once he understands a relationship, he can quickly and correctly replace the rod that is removed from the series. He acquires coordination first by comparing two or three sizes, then by handling five or six sizes. With the help of appropriate hypothetical and convergent questioning he can even deal with a series involving ten sizes. In addition, he can plan this seriation in his head using the concrete shapes as a reference.

The LD child who manifests difficulty with seriation may be having problems with organization, with recognizing patterns or the underlying rule system that generates the pattern. Some of these children can perform seriation tasks by one-to-one comparison using one step at a time which does not require an understanding of the entire situation in advance. Although a LD child might be developmentally ready for seriation, his disorder (for example, his difficulty with organization) presents him with an obstacle to its acquisition. Instruction should be aimed at helping this child recognize the pattern or rule system involved in a sequence.

c. **Conservation of Number:** When mastery of groupings, classification and seriation is taking place, the child is also beginning to compensate internally for external changes in the collections of objects.

Children's conception of number grows out of their direct concrete and visual comparison of quantities. Flavell, in *The Developmental Psychology of Jean Piaget*, says that the whole notion that certain aspects of an object remain invariant in the wake of substantive changes in other attributes comparises the essence of universal thinking.

One of those invariants is conservation of number. To conserve means (i) realizing that changing the physical arrangement of an array of objects does not change the quantity and (ii) being able to hold in mind simultaneously the initial arrangement and its quantity and the transformed array and its quantity. Piaget (1965) believed that the concept of number is reflected in conservation. Even when a child can count, he may not realize that he would arrive at the same number, in whatever order he counted the objects or however the objects were arranged. Binet (1890) was the first to discover the phenomenon of conservation while he investigated number discrimination in young children. Some of his preschool subjects who could not count were nonetheless able to discriminate between 16 and 18 objects, using length of line for their basis of judgement. When Binet altered the length of the lines making the line containing 18 tokens shorter than the one containing 16, his subjects were no longer able to conserve numeration, they judged that the longer of the two lines contained the larger number of objects. Examining various sets of numbers and distances Binet found that 4-year-olds do conserve number if the number of objects to be compared is below five. His 4-year-old subjects could identify as "more" four objects in a row 10 cm long in comparison to three objects in a row 10 cm long in comparison to three objects in a row 12 cm in length. Piaget conducted very similar experiments. He also found that perception of length interferes greatly with cardinal equivalence in the preschool years.

For example you might give the child two equal rows of breads

0 0 0 0 0 0 0
0 0 0 0 0 0 0

and ask if the two rows have equal number of objects in them. If he understands that the number of objects is the same in two rows he will say, 'yes.'

Now, if you push one row together like this

0 0 0 0 0 0 0

0 0 0 0 0 0 0

and ask if there is still the same number of objects in each row, or does one row have more, he may then say that the longer row has more. This means that he is not conserving number, he is merely using the cue of length. Perception is playing a big role in his response at this point. When the child has mastered the concept of conservation he will respond by saying that both rows have the same number of beads, because they both have 7, or if he responds by saying "... because you only rearranged one row, you did not take anything away or add to it."

Once the child has really grasped the concept of conservation he will continue to say that the rows have the same number no matter what you do — spread them out, arrange them in a pattern, or put them into more than one pile. This means that a child realizes that the quantity stays the same unless you add to or subtract something from it, no matter how you change it around as long as only the change in configuration is taking place. This understanding is an essential element in the process of understanding the concept of number.

Piaget (1952) set forth the following three stages for the child's acquisition of this concept:

(i) Absence of conservation (Stage 1). A child is totally unable to ignore his perceptions. He may think that there are more objects in the first collection, perhaps because it is more dense. Or he may think that there are more objects in the two collections, perhaps because they are farther apart.

(ii) Intermediary reactions (Stage 2). This is a transition stage from Stage 1 to State 3. In some cases, the child may give conserving responses (Stage 3), and in others, the child may give non-conserving responses.

(iii) Necessary conservations. A child does not rely just on the information from his sensory perceptions. He knows that the number of objects in the single collection is the same as the number of objects in the two original collections because "none has been taken away or added" or perhaps because "they are the same objects."

Of these stages, Piaget (1964) stated, "The ordering ... is constant and has been found in all the societies studied ... However, although the order of succession is constant, the chronological age of these stages varies a great deal (p. 10)." The ability to conserve number is gained when the child enters the concrete operational stage (about 7 years of age). At this level he can consider two aspects of the situation at once and mentally conserve number.

Robert Cohn (1968) in the study of developmental dyscalculia observed:

> "In various control studies with preschool children about 5 years old, it was observed that some children who could count serially were able to count correctly the number of objects presented in pictures when, and only when, the objects were arranged in a vertical array ... [I]n other groupings, the child was unable to count the number of objects correctly ... The importance of array for correct counting was frequently observed in older children with delayed abilities in language function."

This observation supports Piaget's contention that the development of the concept of conservation of number: 'that number is not changed when a set of objects is partitioned in subgroups' or rearranged into a different configuration is necessary for properly conceptualizing number. Without this sequential number counting is a rote activity.

Conservation of number, thus, **is crucial to the child's understanding of number and indeed to all numerical activity.** From Piaget's point of view, it is the single most important concept which serves as a prerequisite to the understanding of all quantitative operations. A child who has not responded in an adult fashion to the conservation and seriation tasks is unlikely to have abstracted a consistent concept of number, and is still dependent on objects themselves, so it would appear to be unreasonable to expect such a child to manipulate numbers flexibly without reference to such objects (Dickson, Brown & Gibson, 1984).

It makes sense, therefore, to have children start with objects they are familiar with and then use those objects to learn to count them when arranged in varying configurations and in different patterns. Effective teachers never had to have Piaget tell them that children need physical experiences before engaging in more representational and abstract mathematical activities.

Piaget's conservation of number experiments have been much criticized. Some claim they are too 'verbal' and hence children who have the concepts may be misled by the language used (Siegel, 1978). Others contend that the way the situation is arranged predisposes the child to answer incorrectly (Donaldson, 1978; Bryant, 1974). These criticisms are valid to some extent. Nevertheless, Piaget's results themselves have been replicated many times (Elkind, 1964; Dodwell, 1962). Non-Piagetian research, using variants of Piaget's original tasks, has generally replicated his observations, illustrating that (1) children are able to form a one-to-one correspondence before they can conserve number; (2) young children make their evaluations on the basis of relative length, independent of other factors; (3) slightly older children sometimes consider density of arrangement of objects, thus failing to conserve in many situations, and (4) it is not until 7 or 8 years of age that children consistently pass conservation of number tasks (Brainerd, 1977; Pufall, Shaw, & Syrdal-Lasky, 1973).

d. **One-to-One Correspondence (matching)**

A child's ability to match one group with another plays an important role in his conceptualizing number. The idea that two groups can be matched when they have the same number of objects is the key to abstracting the number from its concrete manifestations.

The prerequisite to conservation, therefore, is not counting, but rather one-to-one correspondence. It refers to the child's ability to match the objects in one group against those in another and to make a systematic comparison to decide whether the number of objects in one group is more, less, or equal to the other.

The LD child may have difficulty with one-to-one correspondence in spite of his general developmental level being at the appropriate mental age of 4 or 5 (by this developmental age, a child should have mastered this concept). A likely source of a LD child's difficulty in mastering one-to-one correspondence would be problems in concentration. The larger the amount and the longer the one-to-one matching procedure, the more likely a child with such a learning disorder would be to lose concentration and be unable to manage the amount of incoming information.

A technique to use with such a child might include the teacher's verbal monitoring from the child, "First I put this one with this one, then I put ..." It is good to have the teacher or child use exact descriptors of objects so as not to slip into an automatic recitation and not be aware of what is actually going on. The important part of any intervention and remediation is to maintain active attention to the materials and the concept being demonstrated. A more basic approach is to decrease the demands made on concentration. This can be done simply by using smaller amounts (Cherkes-Julkowski, 1985) and then slowly increasing them to larger amounts.

e. Subitizing and Visual Clustering

Counting is the dominant but not the only way we quantify. Counting is unnecessary for the array of 3 and 4 dots because the eye takes in their "threeness" or "fourness" immediately. This is called **subitizing.** We are also able to recognize what is represnted by ★(one object), through ★★ (two objects), ★★★(three objects), and so on up to about ★★★ ★★★(about six objects) with some degree of reliability. This is an advantage to us because we are able to see at a glance that

★ + ★★ + ★★★ = ★★★★★★.

Arrays of 3 or 4 dots are easily subitized, but as the numbers increase to 9 or 10, this immediate recognition of quantity is not easily possible. Klahr and Wallace (1976) have argued that subitizing is not only faster than counting, but it also develops earlier and is in fact may be a prerequisite to the development of counting in children. Counting objects one at a time works well when the number of objects is relatively small. Where large sets have to be quantified, however, one-by-one counting is not very efficient, and there is a good chance of error. It is easy to lose one's place and either skip one of the objects or count it more than once. Beckwith and Restle (1966) hypothesized that in the case of large numbers, skilled counters first group the objects into perceptually distinct subsets, then either they count or visually observe (subitize) each of the subsets, and finally cumulate the quantities.

However, being able to see more than six objects in a row just by observation becomes unreliable. On the average, people are able to recognize up to about 7 objects by just observation. Beyond ten or twelve, only a few can identify them reliably. Beyond that we have to rely on symbols and counting entirely. But if the same number of objects are arranged in regular clusters such as those found on playing cards, dominos, and other popular objects of these types, then they can easily be identifiable by just observation. This ability is called recognition of **visual clusters** or **visual clustering.** (Sharma, 1981).

Visual clustering ability means — that the child is able to tell or recognize how many objects are there in a collection of objects (when they are in a regular arrangement) by simply observing the collection visually. For example, when a child sees

★ ★   ★ ★ ★   ★ ★ ★
★ ★ ★ or ★ ★ or ★ ★ ★
★ ★   ★ ★   ★

(a cluster of seven objects)

with little practice he is able to recognize (i) that the collections have same numbers of objects in them and (ii) the number of objects is 7. Whereas, if we compare this with the arrangement

★★★★★★★

one would have to sequentially count each time one encounters this arrangement to tell how many there are and the response will take longer.

Recognizing the number of objects by visual observation (Visual clustering or number sense by observation) is quite different to sequential counting. The sequential counting process, although more popular, ontologically, it seems, is of much later vintage, and involves a rather intricate mental process.

To help a LD child, who may be distracted by extraneous information in the problem, it can be helpful to provide redundant cues. That is rather than presenting materials that represent the number 5 with five differently colored blocks or five different objects, use five identical objects. Sets to be arranged in clusters should make the same use of object similarity. Only after the concept of visual clustering in regular arrangements is mastered should sets of different objects to be used in regular and irregular arrangements (Cherkes-Julkowski, 1985).

8. Sequence of Concepts in the Development of Number

a. Counting

Most children acquire the basics of counting well before they actually begin school. For example, when they come to school, they are already familiar with sets of dishes, teams of players, pairs of socks, strings of beads, boxes of crayons, cans of soda, buildings with apartments, their toys as opposed to someone else's toys, and classrooms filled with children. The skillful teacher is able to use children's environmental experiences to help them develop the formal sense of number, counting, and then basic arithmetic operations. Children's self-developed counting procedures are the building blocks of the more complex mathematical concepts that are to follow.

Gelman and Gallistel (1978) have extensively investigated children's early counting and have proposed five principles of counting.

1. The one-one principle: Each item in an array is associated with one and only one number name (such as two).
2. The stable order principle: Number names must be in a stable, respectable order.
3. The cardinal principle: The final number in a series represents the quantity of the set.
4. The abstraction principle: The first three principles can be applied to any array or collection of entities,

physical (for example, chairs, jelly beans) or nonphysical (for example, minds in a room, ideas).

5. The order-irrelevant principle: The order in which things are counted is irrelevant.

The quantification process (sequential counting and visual clustering) can be seen overtly sometimes, but often it proceeds silently. This process is supported by several underlying cognitive structures and processes.

★ Tactile experience: This includes counting physical objects in different configurations by touching, manipulating, rearranging, visually observing and then matching objects to a given value.

★ Audio and visual representation: Along with the tactile manipulation of objects, when the child says the counting words and represents them in written form, he has progressed to the audio and visual representation level.

★ Integration of tactile, visual, and auditory representation: When all are integrated to form the concept of number.

When this integration has taken place there is meaningful comprehension of the quantification process.

b. **Cardinal and Ordinal Numbers:** The number name given to the position of an object in a serial collection is called ordinality and the number of objects in the collection is called the cardinality of the collection. For example: Mark a house with 7 windows from among four choices indicates cardinality and mark the 7th bird in a row of ten demonstrates ordinality.

For Piaget (1952) and many others cardinality involves comparisons between sets. To understand the cardinality principle, children must realize that sets with the same number of objects in them are equal in quantity. One can establish that sets with the same cardinal number are equal through one-to-one correspondence. The concept of cardinal number, therefore, is dependent on one-to-one correspondence and conservation of number. A child truly understands **cardinal number** when he is able to construct a one-to-one correspondence between two sets of objects and is able to conserve this correspondence when it is no longer perceptually obvious.

A child understands **ordinal number** when he is able to form collections in some kind of serial arrangement. To do this a child must first be able to arrange in a sequence a set of objects which differ in some aspect. Secondly, the child must be able to construct a one-to-one correspondence between two sequences of objects in which the elements correspond because they have the same relative positions in the sequences. Thirdly, he must be able to conserve a serial correspondence when it is no longer perceptible. Finally, a child must be able to conserve an ordinal correspondence between two sequences of objects. The conservation of an ordinal correspondence, either mentally or physically, is accomplished when a child can find an object in an unordered set (but a set which is capable of being ordered) which corresponds to a given object in an ordered set.

In understanding the concepts of cardinal and ordinal numbers, the idea of a **unit** is very important. A child cannot construct an accurate one-to-one correspondence between two sets of objects unless he perceives the elements of each set as equivalent units. Similarly, a child cannot construct an accurate sequence unless he perceives each element of the set as a unit which can be put in a sequence because of some characteristic such as length or width, which changes from element to element.

Some authors, for example, Gelman and her group (Gelman & Gallistel 1978; Gelman & Meck 1983, Gelman, Meck, & Merkin 1986) do not pay much heed to any requirements to understand relation between sets. Their criterion is that children should understand that the last number that they produce when counting a set represents the number or the cardinality of the set. But this may not be enough for the complete understanding of cardinal and ordinal numbers. As Fuson (1988) points out, children could grasp that the last number becomes the significant label (cardinality of the set) without realizing why. We have observed that for many children the last number used in counting becomes the name of the last object rather than the cardinality of that set. According to Bryant (1988), one possibility is that children learn to establish equality and inequality between sets by sharing without using number, and that the very success of this strategy prevents them from seeing that number words are another basis of comparing different sets (p. 396).

The concept of cardinality is much easier than the concept of ordinality. **Children,** therefore, **learn the concept of cardinal number before the ordinal number concept.** When questions relating to cardinality and ordinality of a set are asked, most children, between the ages of five and seven give the correct answer to the cardinality questions. Whereas, very few younger children are able to answer the ordinality question correctly. This means that "cardination in counting is attained ... before ordination in counting" (Williams 1977, p. 386). If children are able to give the cardination answer by counting but when tested on correspondence between two sets with same number of objects are not able to give the correct response, then the understanding of number is not genuine.

When children are able to maintain a one-to-one correspondence between the items of two rows after an alteration of the perceptual correspondence between the items has taken place, they genuinely understand the cardinal equivalence of the two rows. If they are not able to maintain a quantitative or cardinal equivalence of two rows of items after an alteration of the perceptual correspondence between the two rows of items then they have not yet acquired the proper understanding of number (Williams 1977; Piaget 1965).

Piaget thus concludes that in order to really understand and apply natural numbers, an appreciation of both ordinal and cardinal aspects is needed, together with the necessary relationship between them. He considers this to be acquired at the same time as many other logical operations develop, at roughly around 6 to 8 years of age for average children. With this coordination, the child acquires the ability to manipulate the numbers in flexible ways to solve problems, for example, by counting on or backwards, rather than totally depending on the presence and ordering of the objects involved. Thus between the ages of 5 and 8 years, the average child begins to develop the ability to reason consistently by reference to numbers, rather than by reference to particular collections of objects.

In some instances, however, it is possible to have ordination response before cardination response. Brainerd (1976), for example, has described how counting can be utilized to generate instances of ordinantion and cardination. He observed that some children were aware of the ordinal names before their cardinal names. This could be regarded as an instance of the ordination-cardination sequence.

Since counting in very young children is well developed only for small sets, a teaching strategy based on this observation will call for allowing children at first to work with very small sets. This is a common teaching approach in most classrooms. On the other hand, if quantification in young children is developed by extending the subitizing skills into the visual clustering skill then one can easily introduce children even to sets with large number of objects very early on. The subitizing and visual clustering skills lighten the processing load on the working memory of the child. Studies of young children's errors in counting show a persistent search for pattern. Of course, the transition from regular arrangement of clusters calls for the mastery of conservation of number by young children. Similarly, since counting appears to become more efficient with age and, presumably, with practice, it may be well to delay conceptual work on large sets until the counting skill with small sets of objects is well developed.

9. Measurement Approach to Developing Number Concepts

Understanding the role of measurement in the formation of initial number concepts has long been a concern of mathematics educators. Most educators think of measurement only as an auxiliary teaching device, but a few of them believe that measurement not only promotes correct mastery of numbers, but supplements this knowledge with the essential elements of relationship, without which numbers in general are not real.

By using the definition of a unit as the relation of a quantity to a unit of measure, Galperin and Georgiev (1969) have outlined a sequence of activities which lead to the formation of valuable elementary number concepts. Measurement is a necessary part of this sequence. The sequence of the process involved in developing the concept of number by the help of measurement is as follows:

- ⋆ Determination of the unit of measure.
- ⋆ Isolation of identical quantities and comparison of similar quantities.
- ⋆ Collection of groups, correlation of groups (one-to-one correspondence), and formation of the concepts more, less, as many as (comparison of groups).
- ⋆ Definition and differentiation of unit.
- ⋆ Formation of new numbers (of the first ten set) empirically (and their initial characteristics, the isolation of first features) by the help of the unit.
- ⋆ Formation of the concept of each number by applying its features in measuring and counting.
- ⋆ Expanding knowledge of each number by correlating it with other numbers.

This group of operations is designed to introduce the idea that the unit of measure is an object in terms of which other objects are measured. The operations include selection and definition of unit and its limitation to a certain type of quantity.

a. Determination of the Unit of Measure

This, strictly mathematical approach to the concept of number, begins with isolating a unit of measure. In a sense it is an extension of the ratio meaning of number as the underlying assumption in this approach is the same.

The concept of the unit has a special place in the formation of number concepts. All other numbers are built from the unit (by the formula n + 1 and n - 1), and even concepts like many-few, more-less, as many as, concepts which at first glance have no numerical content, become strictly mathematical through the comparison of sets by means of a "one-to-one" correspondence. All elementary level mathematical concepts assume the notion of unit. The unit is also a "real" number; it expresses a specific quantitive relationship. Mastery of the initial concept of the unit is the most important step in the formation of elementary arithmetical concepts (they are all built on the unit or presuppose it).

The notion of the unit is best explained by demonstrating individual objects in groups and designating them with the word "one": Here are many blocks (indicate a group of them) and here (select from the group and demonstrate) is one block; etc.; "Give me one puppet," "bring me one cup," etc. By designating different objects with the word "one," the child is taught to pick out this common property from all others which distinguish the objects. Thus, although the definition of the unit is not given, all instruction is designed in such a way that the content of this notion is realized as the individual, separate, object with a single feature — its individuality: "one" is tacitly (inductively) understood as something that is separate or individual.

The above approach deals with unit as a discrete individual object. But many times a unit is not discrete; for example, from the standpoint of set theory, individuality is a characteristic not only of a single object, but of any aggregate of objects which may be considered as an entity. Similarly, the unit of measure: when one object is deliberately isolated as a means of measuring others, any other object equal to it becomes — in this particular relationship to this unit of measure — a unit.

b. Isolation of Identical Quantities and Comparison of Similar Quantities

This group of operations is to illustrate the use of the unit of measure to isolate other identical quantities. It is here that measurement in the strict sense of the word — measurement by number and without it — first comes into play. Several objects and isolated quantities are shown to be equal to the unit of measure, and hence to each other.

c. Comparative Measurement of Quantities

This group of operations is to introduce the comparable measurement of quantities and objects. These operations develop the notion of measurement without numbers which occurs in such concepts as more, less, as many as.

d. Definition and Differentiation of Unit

This group of operations further defines the unit of measure by the isolation of quantities that can be equated to the unit of measure. It is not necessary for a unit to be an entity (not to have parts) or a separate thing (not to be a part of some aggregate of similar units). That means a group of objects can be considered as a unit, i.e., a set (a book can be a unit although it is a set of pages) can be a unit. Assignments like "give me a block" and "bring one book" are rightfully called counting, because they indicate isolation of a quantity equal to a unit of measure and classification of this quantity as a unit.

e. Formation of the Concept of Each Number by Applying its Features in Measuring and Counting

This group of operations in the sequence is concerned with the formation of large numbers from small numbers, particularly from a unit. First of these is the number two. It is formed empirically: 1 and 1 instead of 2. When children learn numbers through rote memory without the empirical process, they do not acquire the actual meaning of a number. For example, when they face large numbers later on, the number means only members of a sequence rather than a definite quantity. To facilitate this, from the beginning a teacher should make use of different objects and ask each time what the unit of measure is. Thus, the correlation with the measure and distinguishing quantity from other concrete properties of the things occurs simultaneously.

10. Children's Number Related Difficulties

Children's delayed acquisition of or difficulty with number concepts (counting, writing of numbers, number relationships) is related to many factors. There can be several reasons for difficulty at this point. The most likely one relates to the learning disabled (LD) child's difficulty in managing large amounts of information at once. To understand any complex concept or relationship, a great deal of information has to come together for a child at a single moment in time. The LD child who tends to be overloaded easily is likely to lose track of the important identifying informations and thereby achieve a faulty notion of cardinality, or just be overwhelmed and confused by multiple characteristics of objects. It becomes necessary to provide special assistance for this kind of child.

Since the learning disabled child is defined as having a disorder in the learning process, a useful model of development for them should include not only an account of the sequence of cognitive steps but also a careful analysis of the procedures that enable a child to move from one set of cognitive forms or schemas to the next (Resnick & Glaser, 1976). It is in the concentration of transitions that teaching is likely to have its greatest effect (Cherkes-Julkowski, 1985). Some of the factors that affect number conceptualization are:

a. Dyscalculia
b. Limited concrete experiences
c. Too much early reliance on language and symbolic activities
d. Lack of integration of visual, auditory, and kinesthetic experiences regarding the number
e. Absence of preskills (grouping, classification, seriation, conservation, and 1-1 correspondence.)

a. Dyscalculia: (For a more indepth understanding of dyscalculia see **Focus On Learning Problems In Mathematics,** Volume **8,** Numbers 3 & 4)

Dyscalculia (other related terms are: acalculia, number aphasia, dysgraphia, etc.) is the term used to describe difficulty in conceptualizing number, number relationships, and determining the outcome of numerical operations. In general this means difficulty in mastering mathematical concepts and/or computations. "A failure to recognize or manipulate number symbols, in an advanced culture, is designated as dyscalculia" (Cohn, 1968). Luriya (1969) considers dyscalculia as "the impairment of the semantic structure of number." He claims:

> "...a disorder in the perception of number never proceeds uniformly; i.e., while some numerical operations are profoundly affected, others remain intact. Thus almost all patients with acalculia observed by us or described in the literature retained completely the ability to perform simple serial computation within the limits of the first ten (and sometimes, considerably further), were able to count easily, could distinguish greater quantity from lesser, and retained completely a knowledge of simple one-place figures and ordinary concrete designations ... Some of them were capable of performing elementary operations with single digit numbers, most commonly addition. At the same time, these patients were not able to perform more complex arithmetical operations, especially beyond the limit of the first ten, and had great difficulty reading, writing, or composing a two-, three-, or multidigit number." (p. 43)

Cohn, like Luriya, identifies the areas of disturbance in the case of dyscalculics at the early level as the failure to develop the ability to recognize number symbols, usually as part of a general language disorder. But at later stages they fail to remember the basic operations, or the use of operator and separator symbols; are unable to recall tables and fail to "carry" numbers in multiplications; and they are unable to maintain proper order of the number in the calculations. Dyscalculia, in cases, includes disturbances in number recognition as well as in arithmetical operations.

Like dyslexia (a child's language related difficulties), according to some researchers, the difficulty, arising as a result of dyscalculia, has been linked to breakdown in some aspects of the functioning or development of the central nervous system. Many others link dyscalculia with severe visual perceptual deficits.

Many children with dyscalculia may excell in reading vocabulary and in syllabication skills. Some of these children have a disturbance in body image. Their drawings of human figures lack organization; details and parts are not structured appropriately. Some of these children show disturbances in visual-motor integration and may have disorientation associated with dyscalculia. Some of these children may not have a strong sense of direction and may have difficulty distinguishing left from right. Often the dyscalculic child is poor in social perception and in making judgements. They have little conception of distance and time. For example, they might have difficulty in estimating the time it might take to finish a given task. On standardized tasks of intelligence, dyscalculic children tend to be considerably higher on verbal than nonverbal functions. But not all children with learning disabilities have difficulty with number concepts. In fact, many children with severe reading disabilities have been observed to be strong in arithmetic skills. Number difficulty (failure to recognize numbers and appropriate operation) could happen without any difficulty with reading and other associated difficulties and vice-versa.

b. Limited Concrete Experiences

Educators are beginning to realize that not all children learn by simple verbal exchange. But there is still little understanding of what concrete operations are and how to proceed from "concrete" to the abstract or how to supplement abstract with concrete or vice-versa. To learn, the child needs to use more than his eyes and ears. A range of visual, auditory, and kinesthetic experiences are necessary for the development of logical and cognitive structures necessary for the conceptualization of number and its use. Churchill in her book, *Counting and Measuring* (1961), emphasizes the importance of concrete experiences. She says that "The messages received by the brain from activity of the interceptors situated in the walls of internal body organs and proprioceptors situated in the muscles and tendons yield information received from the eyes and ears (the extroceptors)." All of this is necessary for learning.

In order to relate the number symbols to the things and events for which they stand, all counting (including large numbers) in the beginning and for an extended period of time should be done by integrating the concrete, pictorial and symbolic activities. With an exclusively symbolic activity, it is difficult to determine the cause of a child's difficulty with number and number operations whether it originated in the conceptual process or just in the recording stage of that operation.

c. Too Much Early Emphasis and Reliance on Language and Symbolic Activities for Number Conceptualization

As mentioned earlier, in some cases, children's mathematical learning processes are affected by delayed language development but exclusive reliance on language to teach number concepts is detrimental to the child. Piaget holds that the sources of thought are to be found not in language, but in preverbal, sensory-motor and concrete actions performed and experienced by the very young child during the first three years of life. The infant, for example, directs his activities first to manipulations. Later he will search for explanations and will want to tell others.

Furth in *Thinking Without Language: Psychological Implications of Deafness* (1963) showed that deaf children have ability equal to normal children when given a test of use of logical symbols, whereas blind children solved the same type of tasks on an average of 4 yars later than did normal children. The reason given is that the children deprived of opportunities for imitation and learning about relationships through actions or objects lack in crucial logico-mathematical operations, whereas the same children do not have much of difficulty in language acquisition and its use.

Language shows a child's knowledge of objects, events, and relationships rather than his reaction to these. In strong contrast to the early emphasis on the development of verbal proficiency, Piaget points out that logical thinking primarily is nonlinguistic, and is derived from first imitating actions and then performing actions.

The social interaction and the resulting language acquisition are important in themselves as they help reduce the child's egocentricity — a characteristic of children mostly in the age group 2 to 7 years. Language has its uses for the child in his development. It is a means of communication — receiving and giving information. Language permits the child to evoke absent situations and liberate himself from the restrictions of the immediate, so as to extend and deepen his understandings. Language alone does not account for the development of the intellect and concept formation.

This proposition is the reverse of the longstanding one in language learnings, that the "word is the thing", that is if we give a child suitable verbal training, he will have the tools with which to think, reason, and to interpret experiences.

These two approaches stand diametrically opposite to each other and as a result do not provide any real help to the classroom teacher, who is trying to help the child to conceptualize numbers and their meanings. The actual situation in learning is different: it is somewhere in between. The child is struggling to conceptualize the number concepts for the first time in his life; it is a difficult task. **A child needs concrete experiences to conceptualize and language to internalize and manifest this learning.** The teachers of the very young need to realize this fact.

The classroom teacher and the tutor need to utilize the interaction of these two approaches so that the learning is a function of acting on the environment and then making meaning out of these experiences. That is, first the child acts on the environment concretely and then through language. He first makes sense out of his environment through actions and then expresses his reactions to these actions symbolically. As a result of all this, learning takes place. So a classroom teacher needs to provide a variety of concrete experiences and at the same time use rich and meaningful language experiences to give meaning to these concrete experiences and record them so that abstractions can take place.

Unfortunately, many teachers and especially teachers who have to deal with large classes, are forced to resort only to verbal or pictorial representations and limited concrete experiences. When children do not understand or grasp the concept presented to them in that form, it is often thought that it is so due to their lack of attention, faulty memory, or something else inherently related to the child. Their assumptions may not be accurate. The child does not understand the concept because he does not simply copy what his eyes have seen or his ears have heard. He needs concrete models to make sense out of the concepts presented. He needs to reconstruct the reality for himself. Purely a linguistic picture and abstract representation does not provide for that reality and as a result the child is not able to comprehend that concept.

In early childhood no efforts should be spared to incorporate concrete manipulative activities in presenting the concept. The child's language at that time is only an expression of what he is thinking, not a way of acquiring thinking.

d. Lack of Integration of Visual, Kinesthetic, and Auditory Form of Number.

For proper and longer lasting understanding of number, children need to learn meaningful counting. Meaningful counting is the first stage of number understanding and it involves the integration of the following activities: physical act of counting; representation of counted result; and identification and recognition of symbols.

When children are involved in these three activities and integrate them, then they begin to acquire the concept of number and counting to any number and eventually with practice master even the idea of large numbers.

11. Special Situations

Due to visual, auditory, and/or tactile problems, many students at least initially seem to have problems with the concept of number. In those situations the teacher has to try special methods. For example:

- more physical experiences before visual representation and auditory recognition of numbers;
- writing a number on the back of child's hand and asking him to match the number of objects with the numeral by actual count or manipulation of physical objects;
- tracing the numerical symbols (numerals);
- cutting numbers from objects of different textures - smooth, rough, different colors, and different materials, etc.;
- making recognizable clusters of grooves in wooden cubes;
- sand and water table experiences (writing numbers in sand and water);
- games and other real life activities.

The key element in teaching number and all other early concepts in arithmetic is that the demonstration using concrete materials produces better understanding of the concepts. The human race has devoted an inordinate amount of time in order to develop and learn the concept of number, mainly through concrete experiences. A child also needs a lot of concrete experiences to conceptualize number and its use (Sharma, 1988).

Since learning disorders can be characterized as problems in attending to the critical feature in a situation, the need to ignore irrelevant information and to focus only on the aspects of number, can prove a problem for LD youngsters. What can make it more difficult for the teacher is the pattern so often described: Sometimes the child seems to know her numbers, the next day she seems to have forgotten the whole thing. She has never learned or has forgotten. At one time, she does focus on the clues that help her discern number. The next time she has switched her focus to something else. She is no longer answering the number question. Her answer may reflect comparisons based on size or shape or color or some other commonality (Cherkes-Julkowski, 1985). The primary goal of instructional assistance is to bring the two critical conceptualizations about classification and seriation together for the child and to make explicit how they relate to the concept of number.

12. Implications For Teaching Number Concepts

The research cited here and our observations illustrate the difficulties a child encounters in fully understanding the concept of number and then using this understanding for solving number related problems. Although the absence of Piagetian concepts does not mean that the child is incapable of learning number concepts or is not ready, these tasks can be used by teachers to assess the child's preparation for number experiences and his progress and then to gain insight into the child's particular view of number. The stages and steps identified should be treated as guidelines rather than hard and fast identifiers of the child's ability in and mastery of number. The key issue is that this information can provide helpful insights for planning learning experiences for the child. The nature and quality of the child's encounter with both planned and unplanned concrete experiences will determine the child's progress in learning number.

For example, a child who has not yet responded in an adult fashion to the conservation and seriation tasks is unlikely to have abstracted a consistent concept of number, but will still be dependent upon the objects themselves, so it would appear to be unreasonable to expect such a child to manipulate numbers flexibly without reference to such objects. Introducing such a child to the abstract form of number without a variety of concrete experiences will result in frustrations and failure which in turn will result in a negative attitude toward learning in general and toward number related activity in particular.

A good way to teach children number and simple arithmetic is to build on their informal knowledge and ethnomathematics. When teachers begin instruction by using children's informal knowledge and then proceed to more complex and formal concepts and operations, children learn readily and enjoy it. This is why learning to count everyday objects is an effective basis for early number and other arithmetic lessons. Parents and teachers can assist by encouraging children simply to count, but further questioning referring to the size of collection, the order of the count, the relative size of two collections, and so on, may help the child to construct the important relationships which need to be made before the process of counting can be usefully applied. This questioning process will help the child abstract the concept from concrete situations and move from specific to general. This is a necessary requirement for learning number meaningfully.

One problem that the young child has, is keeping control of the number of objects involved, while he is counting e.g. which objects he has already counted, remembering to say only one number for each object, and so on. This suggests that early counting experiences should be made as straight forward as possible, involving perhaps a row of solid objects which can be moved aside when counted. Only later when his counting has become more automatic should the child be expected to cope with a random array of fixed objects, such as the pictures which represent concrete objects in early textbooks. Pictures are poor substitutes for concrete experiences.

Considerable practice in the counting process enables children to apply it more automatically, and so one should concentrate on other aspects like those required in accurate enumeration, or making the link between counting and the size of a collection.

Young children are comfortable with numbers; "math anxiety" comes in later years. Just watching the enjoyment children get from songs and nursery rhymes that involve counting is ample evidence of their natural ease. These early counting activities can set the stage for later, more formal exposure to arithmetic. Nursery schools are brimming with activities which are usually classified as play but contain a great deal of mathematics. In fact, more number work is encountered in nursery school than in any other educational setting with children busily engaged in activities described as creative play, imitative play, sand play, water play, model making, carpentry, painting, picture and pattern making in two and three dimensions, collage, block building, clay, dough, plasticine, the home corner, outdoor apparatus, cooking, shopping, weighing, measuring, sewing, bead-threading, paper cutting and folding, books, stories, jingles, etc. (Choat 1978). Although, these activities mainly restricted to physical experience are necessary for mathematics learning, physical experience alone is not sufficient. The child needs to acquire logicomathematical experiences, that come out of these physical experiences as a result of reflecting on and interpreting these experiences. Children need help to translate what they are doing and this translation comes through guided discovery by the teacher and by interpretation through language from the teacher. Words are the first mathematical symbols encountered and learned by children. Words interpret and communicate the first conceptualized spatial relationships. In number, although the "oneness of one," "twoness of two," etc., may be conceptualized, the information must be communicated by symbols (words), e.g., "one," "two," etc. before relational understanding can be ascertained. This translation comes through guided discovery by the teacher and by the interpretation through language from the teacher. Children's vocabulary development, therefore, should proceed alongside physical and logico-mathematical experiences (Choat, 1989). Only a small proportion of children are fortunate enough to attend nursery school so the task of adjusting the majority of children to these learning task rests with kindergarten and first grade teachers. The initial enthusiasm of young children and their subsequent attitude to mathematics can be affected drastically by the manner in which mathematics in general and number conceptualization in particular is viewed by their teacher. This can be either as thought mechanisms through relational understanding when children are encouraged to inquire and come to terms with their environment, or a more formal encounter through instrumental understanding with a subject known as mathematics.

But counting is not limited to merely reciting strings of numbers. It also includes matching numbers to objects and reaching totals (for example, counting the number of apples sitting on the table). Children learn to do arithmetic by first mastering different counting strategies, beginning with rote counting (1,2,3,4), and progressing to memorized computations (2x2 = 4). As children learn the facts of arithmetic, they also learn to combine those facts by using more sophisticated strategies. As their skills grow, they rely less and less on counting.

Summary:

The major development in pre-school years is not the ability to count, but the ability to use the counting process in assessing the sizes of collections. Clearly the ability to count is not acquired spontaneously, but by imitating the action of others. However, the child has to construct for himself the connection between the recitation of arbitrary sounds and the size of the collection.

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MATHEMATICS AND WRITING: SHOWING ALL WORK
WRITING IN THE MATHEMATICS CURRICULUM: AN INTRODUCTION

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As the title of this paper indicates, talking about writing in mathematics involves showing all work. Many mathematics teachers are leery of incorporating writing into our classroom activities because we fear extra work and lack of confidence in our knowledge of the writing process. In this brief paper I shall point out that writing will not necessarily create more work and that we really know more about the writing process than we think.

First, the paper will explore the literature on the topic of writing and mathematics; second, it will discuss some personal experiences; and third, it will offer additional examples that may be explored.

Ten years ago one could find very few articles on the topic of writing in mathematics. Today there are articles in journals, dissertations, presentations at professional meetings like this one*, and books that are all focused on the topic of writing in the mathematics curriculum. Emig (1977), a pioneer in examining that writing is a unique mode of learning, points out that writing allows one to visualize his/her thoughts and therefore examine and modify them. Since writing is active, engaged and personal in nature, it provides a unique mode for self-learning in many academic subjects, including mathematics.

Fulwiler (1980, 1982) built on the work of Emig and others and has been one of the leaders in the movement called "Writing Across the Curriculum." He supports journal writing and expressive writing as two ways of allowing students to reflect on their attitudes and feelings about mathematics. Fulwiler has done much to promote writing in all the disciplines in order to improve knowledge in those disciplines.

Connolly (1989) explains that more recently the ''Writing Across the Curriculum'' phrase has been replaced by "Writing to Learn." He explains that the reason for this change is the following:

> ...because it suggests the powerful role language plays in the production, as well as the presentation, of knowledge. "Writing to Learn" is less about *formal* uses of writing to display memory and test mastery than it is about *informal* writing; about language that is forming meaning; about writing that is done regularly in and out of class to help students acquire a personal ownership of ideas conveyed in lectures and textbooks. (p. 3).

Connolly and Vilardi (1989) have edited a book of essays that examine a range of issues in the "Writing to Learn" movement focusing on mathematics and science. These essays constitute a rich source of material on the topic in one volume.

Johnson (1983) advocates formal writing projects in the mathematics classroom in order to help students in the problem-solving process and stimulate creative thought. He suggests projects like term papers focusing on biographical sketches of mathematicians or tracing the origin of a mathematical concept.

Denmark (1983) describes ways to improve student comprehension of word problems. While this idea focuses on reading, it does tie into the idea of being able to comprehend what one reads in order to write about it. If students can improve reading skills, then writing will be helped.

Milligan and Milligan (1983) discuss what they call "a linguistic approach to learning mathematics vocabulary." This approach employs classroom techniques to analyze vocabulary in order to better understand the meaning of common mathematical words. Students study prefixes, suffixes and roots to obtain a better grasp on mathematical meaning. Again, this notion, which does not specifically address writing, could aid the writing process by helping students clarify their thought process.

Mett (1987) describes how she uses three writing approaches in her calculus class to help students understand the mathematics they are studying. Students in her classes keep journals, and she responds to these journals through open letters to the class. In addition, she has students write in class briefly about mathematical concepts in order to heave them focus their thinking on the day's activities. For honor classes, she recommends formal writing assignments, such as term papers.

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Rose (1988) has explored in depth the relationship between expressive writing and mathematics learning. Exploratory or personal, expressive writing allows the writer to think about a particular subject in order to gain self-knowledge about his or her feelings regarding the subject and insight into the subject itself. Rose used journals, focused in class writing, and autobiographical writing in a calculus class for business majors. The writing allowed Rose to gain insight into her teaching and into the students' ways of thinking about mathematics. In many cases, she had a dialogue with students in their journals in order to clear up misunderstandings.

Williamson and McAndrew (1987) who describe how writing can be integrated into college developmental mathematics classes point out the similarity between writing and problem solving and how each cognitive activity may help students in solving word problems. The authors outline a strategy for teachers to adapt writing into their classrooms that requires a minimum time commitment from instructors. This article may be a starting point for mathematics instructors who are interested in exploring writing as a learning strategy in their classrooms.

The NCTM's standards that were published last year, Romberg (1989), devote a major portion of the document to mathematics as communication and specifically note the usefulness of writing in helping students to think clearly about mathematics. These standards provide a rich resource for teachers who are interested in experimenting with the writing process in their classrooms.

As you can see by the few examples given above, over the past ten years the topic of writing and mathematics has expanded from a few isolated articles to a rich selection of literature that can be used and experimented with by the classroom teacher.

Using Writing in the Mathematics Curriculum at the National Technical Institute for the Deaf

The National Technical Institute for the Deaf (NTID) at the Rochester Institute of Technology prepares post-secondary hearing-impaired students for careers in a wide variety of technical areas. Coming to us with a wide range of language abilities, many students struggle with English and mathematics which are needed to be successful in their career choice. Hearing-impaired students encounter many of the same difficulties that their hearing peers do with mathematics but have the additional problem of being behind their hearing peers in language development.

Over the past four years, a group of department members have developed new mathematics curriculum materials that attempt to use language our students can read and understand while developing a logical and deep comprehension of the mathematical concepts involved. When creating these materials the authors recognized the importance of students writing their responses out in complete sentences. All word problems are required to have a complete sentence for an answer. This activity begins to address some formal aspects of writing in the mathematics curriculum as described above.

Recently at NTID a task force on writing in mathematics was established to assist mathematics faculty who are interested in incorporating writing into their classroom routine. Goals for this activity have been established (see Attachment A) and will be used as a framework to develop projects that will address these goals.

In addition to this work, several of our instructors are studying how the language abilities of our students can be used by teachers in order to enhance students' success in learning mathematics. Faculty have been working with students who enter NTID with both weak mathematics and language skills. The purpose of this study is to identify students who can benefit from strategies that will enable them to strengthen their skills in both mathematics and English.

Our faculty have identified and experimented with several writing activities that help students thinking about mathematical concepts. These activities encompass a range of different levels of courses from introductory to calculus level. In one introductory level course one instructor asked students to imagine they were a fraction and write all about their interesting existence for about a half-page. Another instructor asked students to write a detailed description of how to prove two lines are parallel. On a calculus test one instructor asked students to explain in writing two methods to determine relative maximum and relative minimum points. The same instructor created a co-operative learning activity where a group of students individually worked on problems evaluating limits and studying continuity. The students then got together in groups of 3 or 4 and collectively wrote an explanation to a friend on how to find a derivative.

A written homework assignment at the intermediate level of our courses involved the concept of rates. Students were asked to write for a half-page what rate meant, solve a rate problem, and then create their own verbal problem involving the concept of rate.

These activities represent the creative efforts of many of our staff members who see merit in integrating writing activities into the mathematics curriculum in order to deepen understanding of mathematical concepts.

On a more personal note, I became interested in writing and mathematics about five years ago when I read a striking essay by Baum (1978) who points out that both mathematics and writing are deliberate self-conscious acts, with a shared sense of discipline. When an individual is "doing" mathematics, it is an active process that promotes focused thinking, just as writing.

Keeping this in mind while co-teaching several courses with a colleague of mine, I tried to experiment with some writing activities. In one class when we finished a unit on graphing, I asked students to take about twenty minutes to respond to the following six questions:

1. How will the content of Unit B help you in your major?
2. What did you like best about Unit B?
3. What did you like least about Unit B?
4. How did you like the written materials of Unit B?
5. How did you like the teaching of Unit B?
6. What are you thinking about right now?

I was somewhat apprehensive about this assignment since I had never tried anything like this previously. To my surprise, the students enthusiastically responded; even students who lacked confidence about their writing wrote. After reading each response, I made some general comments on each paper.

In another class my colleague developed a ten minute writing assignment (see Attachment B) which was given to students in class. The assignment was read and commented on, and students were awarded ten points on their unit test for completing the exercise. The writing exercise indicated the students had understanding or lack of understanding about the graphing techniques studied. Several students showed genuine insight while other students disappointed us. Even though the results of this exercise were somewhat discouraging, my colleague and I considered the activity worthwhile for both the students and the teachers.

Additional Examples of Writing in the Mathematics Curriculum

I have just given some examples of writing in the mathematics curriculum. The following examples may also be duplicated or modified in your classrooms.

Students can be asked to keep a journal that will record their daily or weekly experiences and can serve as a method for carrying on a dialogue with individual students. The teacher can comment on student misconceptions and perhaps gain insight into his or her teaching techniques. Also, the teacher should collect and read the journals at specified intervals. Credit can be given to students for keeping these journals or the instructor can make them a requirement of the course. These journals can serve as valuable information for the teacher in planning future lessons. The use of journals is elaborated on in several of the sources mentioned above (see Fulwiler (1980), Mett (1987), Rose (1988), and Romberg (1989)).

On the college level written homework can be used to emphasize the relationship between writing and understanding mathematical concepts. Price (1989) has provided a set of guidelines for applying this technique. This article is a starting point for those mathematics instructors who want to use this idea.

Additional examples of writing that have been described in the literature are letter writing, problem writing, written reports and written explanations of how problems were solved. Let me briefly elaborate on these examples.

Students can be asked to write a letter to a friend describing something that occurred in mathematics class. Such a letter could provide the teacher with insight on planned activities that were successful or not successful. This kind of writing can occur as frequently as the teacher desires.

Another technique which asks students to create their own verbal problems with a solution provides students with an opportunity to be creative and thoughtful about mathematics and the use of language. This exercise was mentioned above. A variation of this theme presents the student with an answer and asks him or her to write a question.

Some teachers have asked students to write reports on a mathematical topic, a famous mathematician, or mathematics influences on our social and political structure. These reports help students to formulate ideas and reflect about the value of mathematics to our society.

An additional technique that helps teachers gain insight into their students' thinking asks students to write about their solution to a particular problem. This exercise aids those teachers who are most interested in their students' cognitive processing.

There are many other ideas that the interested reader can discover from the literature available and from discussion with colleagues. I have shown you that writing in the mathematics curriculum involves showing all work and that it is also an endeavor that will benefit your students and your teaching.

Attachment A

Goals For Writing in the Mathematics Curriculum

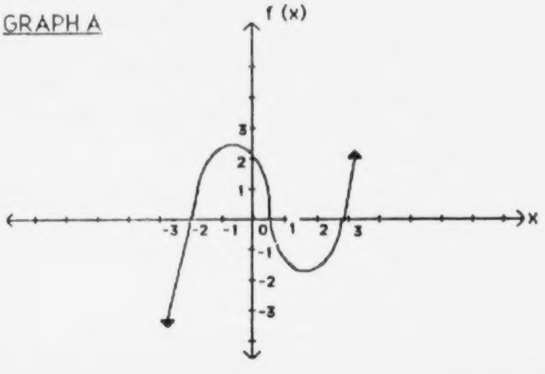
Overall Goal: To enhance understanding of mathematical concepts by helping students develop the capability to use writing as an alternate mode of expression.

1. To assist students in learning to read mathematical writing.
2. To provide feedback to the instructor in order to better assess student background and progress.
   (a) To determine student preconceptions and comprehension of concepts discussed.
   (b) To determine the degree of accuracy with which students use terminology and concepts.
3. To help develop student ability to engage in mathematical dialogue.
4. To promote student metacognitive activities related to mathematics.

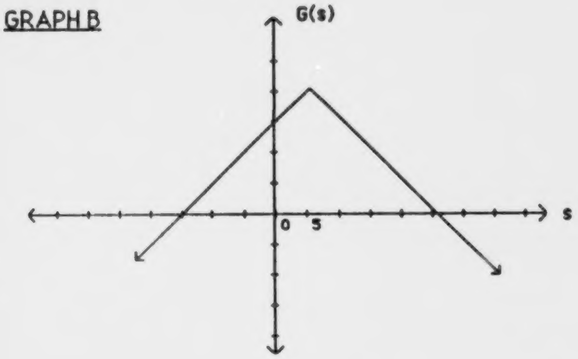
Attachment B

Directions: We would like you to write about the graphs below. You may use English sentences or mathematical symbols or both. KEEP WRITING for ten minutes. Write everything you can think of!!

GRAPH A



GRAPH B



A. Write about Graph A.
B. Write about Graph B.
C. COMPARE Graphs A and B. (What's the same? Different?)

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MATH SELF-HELP VIDEO PROGRAM

By RICK MOORE
GLOUCESTER HIGH SCHOOL

> "I thought it was a good experiment, being in front of a camera and all. It was a good experience. I was nervous but it teaches you how to talk and teach, when someone is there filming and watching you."
>
> Derek Reed, 11th-grader

Not a bad endorsement from a student talking about graphing linear equations, eh?

Derek was one of forty students who took part in the Math Self-Help Video Program at Gloucester High School. The idea of a math self-help video program was conceived while I was participating in the North Shore Writing Project. The Writing Project's goal is to help teachers help their students learn by "writing across the curriculum."

While many courses use writing in a formal-paper sense or make use of journals, there was nothing in the project I felt I could use in math on a regular basis. My students are required to keep notebooks, but they are not required to submit any writing samples (other than proofs and demonstrations of mathematical problem-solving).

If, however, they could somehow integrate their notes into a lesson plan, I felt they might be able to better understand a topic with which they had previously had some difficulty.

So, I developed a five-assignment project which would take them from a general topic to a specific explanation. Each student in Algebra II and Trigonometry selected a topic with which he or she had had difficulty during the first semester. The first assignment was due in two weeks; assignment two was due in three weeks; assignment three was due in three weeks; and it took about three weeks to complete assignments four and five. Each assignment was evaluated and graded promptly. The final evaluation came when each student reviewed his or her tape, as well as one other. So each student evaluated two lessons in all.

As each deadline approached, I eased up on regular classroom homework assignments, while providing extra-help sessions after school. Otherwise, the project was primarily an outside-of-class research endeavor for each student as we continued with the regular course curriculum in each class.

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*The Math Self-Help Video Program is designed with two objectives in mind: the first is to help students understand a topic in mathematics by writing about it; and, the second is to build a library of video tapes which can be used by students whenever they need more help than school-time or after school-time with a teacher can provide.

Each writing assignment, while informal, was carefully scrutinized to determine if the students could explain their topics reasonably well. If not, the students were asked to revise their assignments and re-submit them. The first assignment is to get the student to first recognize and then admit a problem area. Then he or she needs to focus on that problem by trying to articulate it in a topic paragraph. Requiring the student to identify a topic sentence should help him or her zero in on the difficulty.

The following topics were chosen by my algebra students:

1. Multiplication and Division of Algebraic Fractions
2. Graphing Systems of Inequalities
3. Solving Formulas
4. Factoring Quadratic Expressions
5. Graphing Linear Inequalities
6. Solving Systems in 3 Variables Using Matrices
7. Solving Systems in 3 Variables
8. Solving Word Problems Using Two Variables

And my trigonometry students selected the following topics for their videos:

1. Period of Sine or Cosine Curve
2. Sum and Difference Identities for Sine
3. Graphing Sin x and Cos x
4. Linear Interpolation
5. Graphing Sin x, Cos x
6. Derivation and Use of Double-Angle Formula
7. Definition of Trig Functions
8. Bearing

Assignment two is another informal writing assignment which requires detailed research of the student's topic. There are ten questions that I came up with that are specific enough to help the student set up and organize a lesson plan. The questions are as follows:

1. What concept or idea will you try to explain in your lesson? (What is the objective of your lesson?)
2. Are there any specific rules you must review before presenting this lesson? If so, what are they?
3. Are there any definitions or concepts which need to be reviewed or explained before presenting your lesson?
4. What aids do you need to help explain your lesson? (Notes or examples on board; props; overhead projector; co-ordinate system, etc.)
5. What examples of problems will you use to demonstrate the lesson?
6. How will you cover your topic in a step-by-step manner?
7. How will you summarize your lesson clearly and concisely?
8. Can you cite two or three different examples (from your book or any other source) demonstrating the concept you are trying to teach?
9. How can you determine whether or not you have accomplished what you set out to do in your objective?
10. How has this project helped you better understand the topic you selected?

This second assignment forced students to research their topics carefully. Many took the opportunity to think out the teaching process and translated formal definitions into their own words. They also seemed very motivated to tape a lesson with the idea of "helping next year's kids" the prime mover:

> "I understand how to do or solve systems of two equations in two variables I had trouble with before (along with graphing) but now I can do it. Doing the project made me not as afraid to speak aloud (and in front of a camera). It was something a little different from everyday routine lessons.
> It will definitely help the students who take Algebra II next year. We taught the lessons in 'our language' that other kids will understand better. It also helps the person who teaches it."
>
> Kelly Galante, 11th-grader

Most of the comments I made on their assignments were slight corrections in their explanations. Only a few papers needed to be revised and re-submitted.

The detail in researching this paper was evident in most cases. On several papers, I noticed a combination of textbook explanation, my lecture, and the student's own thoughts:

> "Not only did I learn solving formulas better, but I learned things like factoring and multiplying both sides of equations much better..."
>
> Peter Novello, 11th-grader

> "I now understand how to graph systems of inequalities, and before the project I really didn't understand how to do them at all. The project helped me to understand that section and I'm sure everyone else that did it learned too. I hope this project will help the students in Algebra II in the years to come."
>
> Keturah Buck, 11th-grader

I did run into a couple of problems with this second assignment, however. Several students dealt with the mimeographed question sheet as though it were a workbook paper. They answered some questions with "yes" or "no" and tried to "fit in" answers between the lines. At the suggestion of Lynn Furler, the director of the North Shore Writing Project, I asked students to expand and revise this assignment by writing at least two sentences, answering each question. They then re-submitted their papers with somewhat better results.

The third assignment is the actual lesson plan. I tried to leave this assignment open-ended enough so that it would allow the student plenty of leeway; but, I also wanted the student to consolidate the second assignment into a cohesive lesson. I asked them to detail in narrative form how they planned to teach the lesson, including explanations and examples.

While this portion of the project was really nothing more than a review of assignment two, it did allow for more editing and revision before it was submitted as a kind of script or "term paper." Mainly, this assignment provided the student with practice to gain more familiarity with problems, explanations, and examples.

The video taping session is assignment four. The logistics of completing this part of the project took some imagination and plenty of staff support. My Algebra II and Trig class were scheduled first and second periods (in the morning) respectively. For the first week-and-a-half of taping, I tried to teach for half the period, then videotape one student, with another math teacher voluntarily covering my class for the rest of the period.

The only room available for taping was the math department office, a converted classroom. While several teachers were inconvenienced as they worked, typed, corrected, prepared or mimeographed, they were, nonetheless, very supportive. In fact, the support for my project has been overwhelming. From the enthusiastic go-ahead from my principal, Dr. Charles Symonds, to the help from fellow math teachers, I had 100% backing all the way. Manny Martins, the media director at the high school, was instrumental in our taping; the Gloucester School Connection, a local parent organization, provided funding for blank videocassettes; a local video store, The Video Playhouse, gave us a good deal on the cassettes; and, of course, Lynn Furler of North Shore Community College, provided the optimism, enthusiasm, and support to complete the project (which is only the beginning of the program).

The average taping session lasted twenty to thirty minutes, so we often ran over into the next period. Nobody seemed too upset with the process, however, when we modified the taping schedule after the second week. Dr. Symonds allowed me to tape during my duty period (I had cafeteria duty during lunch time, which he covered for me), and I was also able to schedule students during my prep period. With plenty of help from Mr. Martins, who continued to tape for me during first and second periods, I was able to get through most of the rest of the sessions by the end of the third week.

Naturally, the students were nervous at the beginning of each session, but once they got into the lesson, they settled down:

> "The best thing that happened was that I learned the subject I tried to and it 'broke the ice' so that next time I have to do this I know what I need and I will be less afraid of the camera."
>
> Jeff Barkhouse, 11th-grader

> "I was very glad to have been a part of it at the end. In the beginning I never thought it would really happen and was very nervous about it. Although, I understand my topic and feel proud to have done the video because it takes guts and hard work. I'm sure everyone that did it feels that way because everyone worked hard. Also, next year the tape might help oithers."
>
> Becky Hart, 11th-grader

Because of time constraints and logistics, we just let the tape roll without worrying too much about production values. (That can come later.) If a student faltered, we corrected the problem together, and continued the lesson.

Even though I was evaluating students mentally while we were taping, I took careful notes when reviewing their tapes. Trying to be as positive as possible, I looked for various explanations and examples I felt would help future viewers. I also tried to evaluate the performance of the student according to his or her preparation.

Finally, I wrote p a "peer review" (assignment five), so I could compare it with each student's critique.

The results of the project were gratifying: out of forty students who began the project, twenty earned a grade of 91 or better (which converted to an "A" for a double-test grade); ten students earned a "B"; two students got a "D"; two failed as a result of not completing the project; one student received an "incomplete"; and a total of five students dropped out of the courses. We made a total of twenty-five tapes, of which seven are definite keepers for the math video library.

Grades were based on the following table:

Assignment 1 — 10 pts.
Assignment 2 — 30 pts.
Assignment 3 — 50 pts.
Assignment 4 — 50 pts.
Assignment 5 — 10 pts.

A — 135 to 150 pts.
B — 120 to 134 pts.
C — 105 to 119 pts.
D — 98 to 104 pts.

Even more gratifying than the grades were the comments made by students when surveyed on how they felt the project went:

> "I think it was good everybody in the class talked about it outside of class, so it had to be good"
>
> Peter Novello, 11th-grader

> "Gained a further understanding of my subject and now I also understand how much preparation is needed for just a small presentation, like mine."
>
> Chris Veator, 11th-grader

> "I learned how to graph linear equations. I think that it helped everyone that did it to understand and it will help people coming up, who have problems, to better understand."
>
> Chuck Karvelas, 10th-grader

> "I feel that the most important asset attained from my project is that I now know how to solve 'Systems of Equations in Three Variables.' Also I attained more self-respect and confidence from watching my video."
> "I think that the project has many important values, like it is the start of a very helpful video library. Students in the future can also make lessons to help others."
>
> Avery Barry, 11th-grader

> "It let me know how it felt to be up at the board and helped me understand that teaching isn't easy. Plus it feels good that a student next year or whenever can watch my tape, and I can help them. I think it's great that kids could learn from other kids

because you can't always understand the teacher. Then there's the fact that they're on tape and you can pause, rewind, whatever.

Donny Burnham, 12th-grader

"First of all the topic I chose, I had no idea how to complete or solve problems pertaining to it until I wrote out the project. Secondly, I got a feel or better understanding of what it is like to teach something, especially math. Also, the project gave me a little exposure to being filmed and public speaking. It is just an all around positive, constructive learning device that I got a lot out of."

Corey Parsons, 11th-grader

"I think the idea is good, because I know next year if I run into a jam and can just get a video of my Trig plan I'm having a problem with, I will be most gracious."

Dick Lowe, 11th-grader

"I just wish we had tapes for this year, but the idea is certainly very good and students may now enjoy this privilege from next year on. I'm looking forward to Trig. videos next year!"

Becky Hart, 11th-grader

This project has been exciting and successful. I am convinced that students have strengthened themselves in at least one area of their respective disciplines because of their participation in the project.

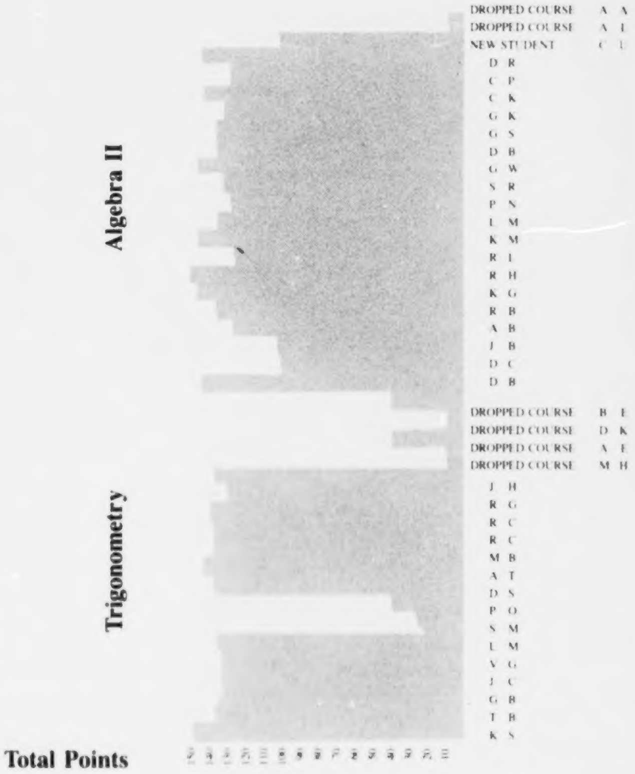
I also see this project as a beginning as far as the use of video technology in math is concerned. Through writing, I plan to continue this program so I can develop a complete math video library for student use. In continuing the program, there are several guidelines I will adhere to in the future.

1. Make the purpose of the student's writing clear to them;
2. Provide them with written copies of the proposal which they will keep in folders available for referral;
3. Prod them constantly, checking their progress;
4. Assign less regular homework as each deadline approaches;
5. Hold mini-conferences with students, while evaluating each assignment;
6. Allow for revision;
7. Grade papers quickly;
8. Evaluate in as positive a manner as possible;
9. Keep a portfolio for each student.

The success of the Math Self-Help Video Program emphasizes how various educational methods can be integrated to develop a new and exciting approach to learning. Hopefully, the program will inspire teachers in other disciplines to come up with their own ideas on how they might use video technology to enhance learning and supplement teaching.

The idea of a math video library wherein a student could check out a tape and view it (over and over, if necessary) has been well-received by many teachers. They have enthusiastically added their own ideas on how to expand on the concept and add further uses to the implementation of video technology.

In summary, it's been a blast!



NORTH SHORE WRITING PROJECT
MATHEMATICS
GLOUCESTER HIGH SCHOOL

ASSIGNMENT 1 (10 pts.)
DUE: JAN. 4, 1988

NAME:____________
COURSE:____________

1. Please detail your first choice for your video lesson (chapter-lesson, etc.).

2. Please detail your second choice for your video lesson.

3. Write a paragraph describing how you will explain your lesson. Underline one sentence that you think describes the main idea of what you will try to explain.

NORTH SHORE WRITING PROJECT
MATHEMATICS
GLOUCESTER HIGH SCHOOL

ASSIGNMENT 2 (30 pts.)
Due:

NAME:__________
COURSE:__________

1. What concept of idea will you try to explain in your lesson? (What is the objective of your lesson?)

2. Are there any specific rules you must review before presenting this lesson? If so, what are they?

3. Are there any definitions or concepts which need to be reviewed or explained before presenting your lesson?

4. What aids do you need to help explain your lesson? (Notes or examples on board; props; overhead projector; co-ordinate system; etc.)

5. What examples of problems will you use to demonstrate the lesson?

6. How will you cover your topic in a step-by-step manner?

7. How will you summarize your lesson clearly and concisely?

8. Can you cite two or three different examples for your audience to try (from your book or any other source) demonstrating the concept you are trying to teach?

9. How can you determine whether or not you have accomplished what you set out to do in your objective?

10. How has this project helped you better understand the topic you selected?

NORTH SHORE WRITING PROJECT
MATHEMATICS
GLOUCESTER HIGH SCHOOL

ASSIGNMENT 3 (50 pts.)
Due:

NAME:__________
COURSE:__________

1. What is the objective of your project?

2. Please detail in narrative form how you will teach your lesson. Include explanations and examples. Use another sheet of paper, if necessary.

3. Please detail how you will determine whether or not you have achieved your objective.

NORTH SHORE WRITING PROJECT
MATHEMATICS
GLOUCESTER HIGH SCHOOL

ASSIGNMENT 4 (50 pts.)
Due:

NAME:__________
COURSE:__________

EVALUATION

1. TOPIC:

2. EXPLANATION OF TOPIC:

3. EXAMPLES:

4. REVIEW:

5. FINAL EVALUATION:

NORTH SHORE WRITING PROJECT
MATHEMATICS
GLOUCESTER HIGH SCHOOL

ASSIGNMENT 5 (10 pts.)
Due:

NAME:__________
COURSE:__________

PEER REVIEW

1. Topic:

2. Explanation of topic
   What suggestions do you have to improve the way this lesson was explained?

3. Examples
   What suggestions do you have for better use of examples?

4. Review
   What suggestions do you have for helping the instructor create a better review?

5. Evaluation
   What suggestions do you have for helping this instructor evaluate his or her lesson? On a scale of 1 to 10 (with 10 being a perfect lesson) how would you rate this lesson?

MATH NOTEBOOK
From Theory to Practice
Information For Teachers/Parents of Children with Learning Problems in Mathematics

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DYSLEXIA, DYSCALCULIA, AND SOME REMEDIAL PERSPECTIVES FOR MATHEMATICS LEARNING PROBLEMS

MAHESH SHARMA

In this article, we will examine the nature of mathematics learning; describe types of mathematics learning problems that students exhibit; explore possible causes of these problems; and then suggest some strategies that parents and teachers can use in order to help children learn mathematics better.

The types of arithmetical and mathematical difficulties that children with or without learning disabilities experience, and how to diagnose and remedy these problems, is just beginning to receive attention from special needs and mathematics educators. We are beginning to see research reports on diagnosis and remediation of children's mathematics learning problems in the professional literature. Teachers working in this area and parents of children with learning problems in mathematics find that there are still fewer assessment devices and remedial materials available for arithmetic and mathematics problems such as dyscalculia than for problems in reading, writing and spelling. Some of the reasons for this are: the widespread concern with literacy; considerable emphasis placed on language arts skills in the primary grades; the lack of emphasis on arithmetic; and the feeling on the part of some teachers and parents that arithmetic is not as vital to future success and preparation for careers as other academic subjects.

I. Mathematics Learning Problems

Many children have problems in learning — many exhibit learning problems only in the area of mathematics. The nature of their problems varies. Some students lack accuracy; while others may be very accurate but compute problems unusually slowly; others have inconsistency in performance — solving problems correctly one day, but not the next; and still many others fail to maintain learned arithmetic and mathematics skills for extended periods of time or have difficulty in applying their knowledge in new situations, whether in other areas of mathematics or another discipline. Many others have developed distaste for and fear of anything mathematical.

As learning disabilities can exist across the intellectual spectrum, the group of children having difficulty in mathematics is also a heterogeneous group of individuals, whose underlying problems, while different, may result in similar as well as different patterns of underachievement in mathematics.

The pattern of difficulties may look similar because most of these students cannot perform the right algorithms and can not recall the appropriate facts in operational procedures.

On the other hand, the mathematics learning difficulties that children exhibit may also indicate different patterns of errors. Their problem may look different because of their different types of wrong answers; the wrong procedures that they have applied in apparently similar situations; and/or the wrong facts that they have given. As a result, the affect of their failures on their learning ability, attitude, and capacity for mathematics learning may also be strikingly different. The impact of any failure and, therefore, the effects of failures in mathematics on the self-esteem of different students is also different.

For example, the student who cannot recall arithmetic facts at age nine and the student who has to devote an enormous amount of time to figure out even the simplest of facts at age fifteen have similarity in the manifestation of the problem from the performance point of view, but both of them may have, because of their problems, developed quite different attitudes toward mathematics. The nature and the course of remediation to be followed in these two different types of cases may be similar in content but quite different in the selection of instructional materials, sequence of tasks, and in the approach to instruction.

The failures which some children and adults exprience in arithmetic, sometimes, are at variance with their basic intellectual ability. For example, a student may experience difficulty in learning arithmetic and/or mathematics in spite of potentially high level of intellectual ability.

Some students are quite capable of learning basic mechanical arithmetic but they may have difficulty in conceptualizing higher mathematics. On the other hand, some children in spite of their difficulty in learning arithmetic are able to learn higher mathematics.

Thus, arithmetic and mathematics learning problems are manifested quite differently by children and many a times carried over to adulthood.

A. **Diverse Perspectives**

The causes of problems in arithmetic and mathematics are also many. The causative factors considered for the explanations of the incidence of mathematics learning problems reflect the different perspectives that observers from various disciplines use in studying these problems. For example, most neurologists explain the mathematics learning problems (particularly dyscalculia and acalculia) in terms of deficits of the higher cortical functioning of the brain (ranging from very subtle to gross). The approach is similar to explaining the incidence of dyslexia and other language related learning disabilities. According to them, any dysfunction — a lesion in some specific part of the brain, an insult or injury to the brain has a potential involvement with learning and, therefore, may become a cause of learning problems in mathematics. In many cases this may be true. Yet, many children exhibit difficulty in learning mathematics for a variety of other reasons and many times quite independently of any discernible neurological deficit or anomaly.

Many of the problems children experience in learning may have neurological basis, as many researchers do attribute certain reading disabilities to neuropsychological factors. Several studies have found that there is a high incidence of failure to achieve in arithmetic in samples of students classified as learning disabled (Fleischner, Garnett and Shepherd, 1982; Goodstein and Kahn, 1974; Sharma, 1979). These studies show that there exist generalized deficits in computational skills, and particular difficulty in the rapid recall of basic facts in students with learning disabilities or dyslexia in early school years (Miles, 1975).

The cognitivists find the lack of cognitive development and deficits in information processing capability of the individual as possible reasons for his difficulties in learning mathematics. These include problems in (a) developing effective memorization skills, (b) acquiring cognitive strategies to represent concrete experiences into abstract forms, (c) generalizing and extrapolating specific experiences to form conjectures about quantitative/numerical information and observe patterns in mathematics concepts, (d) interpreting the given information in the context of the problem, and (5) articulating strategies necessary to solve the problem, etc. Accordingly, from this perspective any cognitive lag may contribute to difficulty in learning mathematics. For example, a student may give the answer to the problem 5894 ÷ 23 if we let him perform the actual calculations procedurally (applying the standard division algorithm). However, if you ask him what do you expect as the answer to this problem he has difficulty giving you the estimated answer. Or if you ask a student what is 1/2 of 8 he can give you the answer as 4 but when you ask him what has he done to get the answer he will answer that he has divided 8 by 1/2. In these cases, as you can see he has found the right answers to the problems but he lacks conceptual understanding. In such cases, not only he lacks conceptual understanding, but he has difficulty in generalizing the result to other related problems. A child with this difficulty is generally good in basic procedural numerical computation but poor in mathematical and logical reasoning.

The behaviorists, on the other hand, attribute pupils' mathematical difficulties to such factors as failure to master prerequisite skills, lack of instruction, inadequate understanding of the stimulus and lack of control in problem variables, insufficient reinforcement of learned skills and limited opportunities for practice of the learned skills and procedures, and use of insufficient and ineffective procedures in learning skills including improper sequencing of skill hierarchies and conceptual links. From this perspective, therefore, contributing factors to the incidence of the problem lie outside the child.

Special needs educators, generally, posit that the learning disability imposes limitations on the child's mathematical abilities. As a result, the nature of the learning disability provides the explanations for the learning problems in mathematics.

Mathematics educators look at the same problem from the perspective mathematics content. The lack of preparation for mathematics experiences, poor teaching, lack of appropriate conceptual models, and the difficult nature of mathematics concepts as the reasons for a student's difficulty in mathematics learning.

The complexity of mathematical learning problems shows that the etiology of these problems is as complex and diverse as learning disabilities in general. No one perspective is adequate to fully explain the nature of mathematics learning problems. Therefore, there is no one approach that can fully explain a child's learning problems in mathematics.

There may be several reasons why a child may have difficulty in learning mathematics: (1) mental deficiency and/or cognitive lag in thinking, (2) reading and other related language disabilities, (3) lack of adequate mathematical experiences and/or prerequisite skills necessary for the development of concepts and procedures, (4) emotional and other psychological and neuropsychological factors, and (5) lack of matching in instructional approaches and the child's learning style and needs. The complex and difficult nature of the discipline of mathematics is another important factor which causes difficulty for many individuals.

Kosc (1981) specified four major factors affecting mathematical ability:

(1) psychological factor such as cognitive style and intellectual attributes;

(2) educational effects of appropriate and inappropriate instructional practices and curriculum patterns;

(3) personality factors that influence learning styles and performance capabilities; and

(4) the individual neuropsychological pattern or developmental status (p. 19).

II. Types of Mathematics Problems

As the causes of mathematics problems are many, there is no one thing that can explain a child's learning problems in mathematics nor is there one corrective technique. One has to examine the individual and his problems. One should examine the nature of the problem, its possible causes, and then determine the course of remediation reflecting this diagnostic information.

Some individuals may experience difficulty in learning mathematics because of the limitations imposed on their learning ability and capacity because of dyscalculia and lag in quantitative thinking. For example, every child has a limited capacity for processing information, but the LD child is likely to have even greater restrictions on his information processing capability and/or attention span. More that is brand new and without a frame of reference in a learning situation, the more of a burden on the child's capacity to learn and make sense out of it. Learning disabled students, as a result, have particular difficulty in the rapid recall of basic arithmetic facts. In our clinical work with children, we have found that one of the major difficulty most of these children have is slow or no recall of arithmetic facts. Most of them construct these facts using inefficient and primitive methods such as serial counting.

Some others experience difficulty in learning mathematics because of carryover from a specific learning difficulty such as dyslexia. Still others may have difficulty in conceptualizing mathematical ideas for reasons quite apart from themselves, such as: poor teaching, inappropriate expectations, or improper diagnosis of the problem. These might be quite independent of any discernible neurological problems in general.

Although, some of the arithmetic and mathematics problems are language and reading related, particularly those skills which involve similar kinds of underlying skills, but some learning difficulties are unique to arithmetic and mathematics.

Mathematics is mostly cumulative in nature, as a result, there is a strong notion in the minds of many students, both adults and children, that since they could not do arithmetic calculations accurately they cannot learn higher mathematics either. That is not always true. For example, some children in spite of their difficulty in learning arithmetic are able to learn geometry and/or algebra and vice-a-versa. It should be made clear to the student (whether a child or an adult) that many of the higher mathematics skills are quite different from mechanical arithmetical calculations. It is, therefore, important that the student does not carry his/her feelings of failure in arithmetic to mathematics. It is not enough, however, merely to tell him this. By effective teaching, we should be able to show this to the student that many concepts in arithmetic and mathematics are independent of each other. The key element is that the student needs to taste success in some aspect of mathematics.

As we have seen in the previous sections, there may be several reasons why a child may have difficulty in learning mathematics. Mathematics problems are usually manifested in the forms of *carryover problems, dyscalculia, mathematics anxiety* and/or *lag in quantitative thinking*, regardless of whether the individual has specific learning disabilities or not.

A. Carryover Problems

This category includes those learning problems and difficulties which the child exhibits in the area of mathematics but they are contributed to by problems in other areas such as: reading, writing, psychomotor, spatial orientation, directionality, following instructions, auditory and visual memory, social and emotional maturity, etc. For example:

★ A student with dysgraphia writes something and reads that written information as something else. Many bright dyslexic children have been misjudged because their teachers could not read their written responses. In their arithmetic procedural work, they may not align arrays of numbers in whole, decimals, and fraction forms. The work of extremely dysgraphic children actually resembles "chicken scratching," with few recognizable letter, number or word forms on the page. Frequently they can read their own handwriting, although no one else can. Such a student may begin solution and his related work in the middle of the page and it may be so disorganized that it may be difficult to make sense out of the "mess."

★ A child is not able to master the multiplication or division algorithms because he has problems in remembering the sequence of directions involved to complete the particular algorithm. Experiencing difficulty in orientation of activities such as north or west direction, left and right, up and down, etc.

★ A child confuses between 6 and 9, 2 and 5, + and × etc. because of certain perceptual handicaps.

★ A child draws a triangle when asked to draw a rectangle and vice-a-versa because he confuses in the meaning of words ending with the same sound.

★ Inability in translating mathematical word problems into mathematical operations due to problems in reading comprehension.

★ Lack in organization and coordination skills: unable to manipulate concrete and three dimensional objects, (although good in dealing with two dimensional aspect) thus difficulty in arithmetical operations at the concrete level — counting, grouping, etc.

★ Inability in visual spatial relationships, difficulty in visualizing clusters of objects for estimation, and inability to read and use maps and graphs and has problems with shape and form.

These are some examples of carryover problems in arithmetic and mathematics. As one can see carryover problems originate in other areas of learning, but affect the learning functions in mathematics. In other words, the student is underachieving n the area of mathematics because of his difficulty in areas outside of mathematics.

1. Mathematics Problems due to General Language Difficulties

Learning problems in mathematics may be found with or without student's reading, spelling and other language related difficulties. But, some of the arithmetic and mathematics problems are language and reading related, particularly those skills which involve similar kinds of underlying activities. However, some learning difficulties are unique to arithmetic and mathematics. Some of the same intellectual, physical, social, and emotional factors that contribute to difficulties in other academic areas may, sometimes, be responsible for difficulties in arithmetic in particular and mathematics learning in general. For example,

> If a child has some of the problems of reading, spelling, sequencing and orientation characteristics of dyslexia, it is possible that he will also have difficulty with arithmetic. Indeed in some cases special coaching at arithmetic seems quite as necessary as special coaching at reading and spelling. (Miles & Miles 1975, p. 51)

Mathematics is a form of language and any language development deficit may affect the learning of mathematical symbols and operations. It is also a very highly specialized and abbreviated language; a highly technical jargon. Numbers have no substance and mathematical data are arbitrary sounds or marks called symbols. "Behind these symbols lie the boldest, purest, coolest abstractions mankind has ever made" (Langer, 1957, p.21). The difficulty in symbol interpretation may be one of the reasons for children's mathematics difficulty.

The following are some of main types of difficulties in which failure in mathematics is coupled with reading disability. These are:

(a) inability to read problems, therefore, manifesting difficulties in mathematics word problems,

(b) lack of information concerning mathematical facts due to failure of the child to make normal school progress. Since, the child with reading problem is, often taken out of a normal mathematics class and placed in special classes where the emphasis is on reading progress, therefore, the child may not get enough instruction in mathematics and, thus, may have limited exposure to mathematics, and,

(c) the emotional blocking to learning originally due to reading disability but eventually extending to mathematics also. This may be developed either into a general anxiety about learning and therefore for mathematics or just specific mathematics anxiety. This can also develop characteristics that make it difficult for the child to adapt to the demands of learning situations.

2. Auditory Receptive Language Disorders and Arithmetic

The child with verbal comprehension disorders has difficulty understanding the words to describe certain processes or in grasping word meanings in story problems. Numerical symbols have more stable referent than other verbal symbols. In contrast, spoken words often vary in meaning and are more confusing. Many words used in describing mathematical processes (e.g., set, times, base, rate, union) are unusually difficult for this type of child because he cannot shift meanings from one context to another. He does well in computations but is inferior in reasoning, word problem solving tasks, and on arithmetic vocabulary tests. For example: when dictated

$$48.3 \div 24.82$$

the student writes the problem as

$$48.3\overline{)24.82}$$

and reads this problem as forty eight point three divided by twenty four and eighty two hundredths.

3. Auditory Memory and Arithmetic

A child with auditory memory problems has difficulty in quickly recalling numbers. He recognizes the correct number when he hears it but cannot always say the one he wants. A child with deficits in auditory span has difficulty when the arithmetic problems are presented orally. He cannot hold and assimilate all the facts in his mind so he cannot work the problem. If, however, he is permitted to read or visualize the problem he has less difficulty. These children become overly dependent on written mathematics. Even simple problems involving recall of facts are written and procedures are used.

Grace Fernald says in her book, **Remediation in School Subjects,** "that most of our cases of reading disability think in words or in kinesthetic terms. Such individuals are particularly disturbed by being talked to while they are trying to think. Verbal explanations, while these individuals are attempting to solve problems, are not only hard for them to follow but interfere with the adaptations they would otherwise make. The result is that the lengthy verbal explanations still used in some of our schools block the understanding that would otherwise develop as a result of adaptations to numerical situations. Since the technique used in all our remedial work in mathematics stresses concept development through concrete experience, it is particularly well adapted to the typical case of reading disability" (Fernald 1943, p.162).

4. Sequential Directions and Arithmetic Disorders

Many of the problems (multiplication, division, work with fractions) involve sets of instructions and if a child has problems in remembering or following instructions then he will have problems in those operations. Many LD children show deficits in sequencing; such a deficits has the potential of affecting the procedural aspects of arithmetic and mathematics.

5. Motor Problems and Difficulties in Learning Geometry

Not all poor handwriting and graphing problems are the results of dysgraphia. Those children who have poor eye-hand coordination, and/or poor spatial orientation generally have problems in learning Euclidean and projective geometry concepts. I have seen secondary students who have great difficulty in copying a geometrical figure from black-board or a textbook page to their workbook because of motor coordination, visual perception and spatial-orientation difficulties.

Many children are deficient in visual-spatial organization and non-verbal integration. They cannot quickly distinguish differences in shapes, sizes, amounts, or lengths. They cannot look at groups of objects and tell which contains the greater amount. Some have difficulty estimating distances and making judgments related to visual-spatial organizations.

6. Dyslexia and Difficulty in Mathematics Learning

Many children with reading problems may not have problems in acquiring mathematical concepts. However, they may not have the facility to sound out the words properly; hence, they would not be able to solve problems in which written words rather than the numerical symbols were used. Visual perception usually affects the number work (confusions between 3 and 8 or 6 and 9). One can find examples of rotations, inversions, and distortions in the work sample of these types of children. A disturbance in the revisualization has marked effect on written calculations because the child cannot remember the order of appearance of specific numerals in a number or of the numbers in a problem. The child can copy and reauditorize numbers but he has difficulty in reproducing the numbers and operational symbols in the written form from memory.

Many dyslexic children have problems with arithmetic. In some cases, the very same difficulties that the child experiences in reading and other language concepts, begin to interfere in learning mathematics. Some children with these problems begin to have difficulties in learning even the simplest of arithmetic concepts such as: number relationships, basic number facts. It is, therefore, important to understand the nature of the language and reading related problems to make sense of a child's arithmetic problems.

> "If a child of ordinary ability is late at learning to read and has special difficulty over spelling, and if, in addition, he shows confusion over getting things into spatial and temporal order, one should regard him as having distinctive disability requiring special attention ... The commonly accepted name for this disability (or group of disabilities) is dyslexia" (Miles & Miles 1975, p. 1).

The nature of the reading and language disability also sheds light on child's mathematics learning, for example:

> "There is general (though not universal) agreement as follows: some children are late in learning to read and have persistent difficulty in remembering spellings in contexts where this cannot be due to lack of intelligence or opportunity. There is often evidence of other difficulties — long struggles over learning to tell time (on non-digital clock), uncertainty over left and right, confusion over times and dates, inability to recite, without stumbling, any but the easiest arithmetic tables..." (Miles, 1988, p. 97).

This of course, refers to the child's reading and language problems. But skills such as spatial and temporal order are also important for the conceptualization of arithmetic and mathematics concepts. Because of these and many other underlying similarities in skills in both arithmetic and language, dyslexic students may have problems in number conceptualization. Although the primary problem of the dyslexic child is in language, reading, and spelling the child sometimes manifests problems in arithmetic that are basically of a secondary nature. That is, the child is being provided extra instructional experiences in language areas at the cost of limited experiences in arithmetic, and for no other apparent reason, the child may do poorly. Consequently, in many cases, once the child's primary problem in reading and other language areas is remedied, he may begin to do well in arithmetic, unless, by now, the failure in arithmetic has lowered child's self-esteem. In such a case, we will have to significantly improve the child's self-esteem so that he may begin to participate in mathematics related experiences and may begin to do better.

Along with reading and spelling difficulties, the dyslexic child may typically have problems with arithmetic, such as tables, basic operations, and place value. In some cases, however, the child may have good mathematical ability, in the sense that he may understand concepts in mathematics quite well; it is manipulation of number symbols that is his problem. (Thomson 1984) It is important to differentiate between mathematics on the one hand and arithmetic (or calculation) on the other. Many dyslexics "may quite often be able to grasp abstract and complicated mathematical ideas and yet have difficulty over what seem to the rest of us to be simple calculations" (Miles & Miles 1983).

> "...Just as it is possible to build lexical entries for words and world-parts as well as for single letters, so, too, it seems reasonable to suppose that there are lexical entries which represent not only numbers but combinations and numbers. Thus the typical non-dyslexic adult knows that 52 − 9 = 43 without having to work it out, whereas he does not know, unless he is very exceptional, that 323 x 17 = 5491; and in general we are all aware intuitively of the difference between situations

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when we can instantly give the answer to a calculation sum and situations where we need to work this out" (Miles 1983).

According to Miles' "lexical entry theory, one would expect a lesser range of "immediate knowledge" on the part of a dyslexic person. As a result, some dyslexics may not remember many ordinary facts by memory. We feel that the idea that the dyslexics have "an inability to retain complex information over time" (Miles & Wheeler (1974, 1987) comes also in the way of learning mathematics normally. "They have fewer number facts available to them than do non-dyslexics: thus, if the question is, "What is 6 x 7?" a non-dyslexic of suitable age and ability can respond "42" in one response, whereas many dyslexics can reach the answer only by working it out" (Miles 1989). According to Miles, major problem dyslexics face is that of labeling. Problems over labelling play a significant part in a dyslexic's learning mathematics also: The non-dyslexic will learn, for instance, that 8 x 7 = 56 after a relatively small number of "exposures", whereas the dyslexic, because of his slowness at naming will "miss out" and be unable to make use of the learning opportunities which are presented. As a result, these people need more "exposures" to stimuli in the environment before these stimuli take on symbolic significance, and this delay results in longer "naming time."

The longer "naming time" then has consequences at recall of digits and weakness over calculation. This is evident from the following study by Miles (1987). When 15 dyslexic boys, aged 12 to 14 years, and 15 suitably matched controls were asked the products of all numbers between 2 and 16 the dyslexics showed themselves to have consistently fewer number facts available, particularly in the less "patterned" tables (i.e., those other than 5, 10 and 11). Since there were 15 tables and 15 products per table, the maximum number of "number facts" for each table was also 225. Table 1 shows the difference table by table, between the 15 dyslexics and the 15 controls in respect of "number facts known".

Table 1: Comparison of dyslexics and controls in respect of number facts known on each of the 15 tables

Table	Number Facts	
	Dyslexics	Controls
2 times	164	202
3 times	105	169
4 times	85	151
5 times	118	158
6 times	73	130
7 times	73	126
8 times	69	132
9 times	63	123
10 times	183	195
11 times	126	144
12 times	44	103
13 times	9	27
14 times	9	22
15 times	13	39
16 times	9	26

Table 2 Subtraction of fractions by a dyslexic girl

$$\frac{5}{6} - \frac{1}{3} = \frac{2}{12} - \frac{5}{12} = \frac{3}{12}$$

$$\frac{7}{8} - \frac{3}{4} = \frac{3}{16} - \frac{5}{16} = \frac{2}{16}$$

$$\frac{8}{9} - \frac{1}{2} = \frac{6}{18} - \frac{8}{18} = \frac{2}{18}$$

> The examples in Table 2 were taken from the exercise book of a girl in Auckland, New Zealand, aged 13. Note that her performance is totally logical. The rules, admittedly, are not the rules for subtraction of fractions that you and I were taught, but her behavior is rule-governed and not random.
>
> (1) She knows that one has to look for a common denominator and in the first two examples the choice of 12 and 16 is at least logical.
>
> (2) She appears to be muddling subtraction with cross multiplication; thus 5 − 3 = 2 and 6 − 1 = 5. She then confuses 2 − 5 with 5 − 2.
>
> As you can see that in the second and third examples she is applying precisely the same rules. That she can produce a consistent performance is itself an indicator of good reasoning ability. (Miles 1987, p. 176)

The extent to which number-sense and mathematical conceptualization are impaired and these facts (lexical entries) and appropriate procedures are missing can be expected to vary considerably between one dyslexic individual to another, since the amounts of compensatory early training are likely to be different. According to Miles & Miles (1983) the three most common problems of dyslexics in arithmetic are: (a) difficulty over simple addition and subtraction, (b) difficulty in learning tables by rote, and (c) difficulties arising from uncertainty over direction both in space and time. As many dyslexics may have sequencing difficulty, memorizing the order in which operations are to be performed may not be easy. There may also be difficulty over the 'greater than' and 'less than' symbols and, in more advanced work, over symbols which express direction of flow.

It seems then, that some poor readers are also below average in arithmetic ability, although the precise nature of this association is not clear. There is a fair amount of evidence that arithmetic skills are impaired in retarded readers, although the exact significance of this is not clear. They frequently perform below average on the arithmetic subtests of **Weschler Intelligence Scale for Children** (WISC) and in fact Newton **et al.** (1979) present results of 11 studies which support this finding.

This poor performance in arithmetic in dyslexics could be attributable to some underlying dysfunction such as defective symbolization, a deficit in lexical encoding (Miles &

Ellis 1981) or cognitive and spatial difficulties or a brain dysfunction (Farnham-Diggory 1978). Ramma (quoted in Miles 1987) showed that dyslexics aged between 7 and 9 years were no different from age-matched controls in tasks which did not require labeling (tasks of visual and auditory discrimination) but they were appreciably weaker than the controls in word analysis tasks, and in some of these tasks they were weaker even than the poor readers believed not to be dyslexics. Numerals, of course, no less than letters, have to be interpreted as symbols, and therefore dyslexic persons often tend to have difficulty with basic arithmetic.

> "Dyslexia, which affects three to four boys to every girl, has been associated with slow speech development, speech and language difficulties, delay in motor development and visual perceptual deficits. Other associated factors are difficult to define but include sequential problems (usually the inability to remember the days of the week or the months of year is noticeable) and impairments in temporal or spatial awareness" (Snowling, 1987, p. 2).

These characteristics, of course, refer to the child's reading and language problems. But skills such as spatial and temporal order are also important for the conceptualization of arithmetic and mathematics concepts. Perceptual mechanisms in the discrimination of quantities, language processing in counting and problem solving, decoding of mathematical notational systems, and procedural knowledge in calculation have commonalities with skills needed to facilitate reading and language development. Because of these and many other underlying similarities in skills in both arithmetic and language, dyslexic students may begin to manifest problems in number conceptualization. For example, relationship between reading and short-term memory or the articulatory loop of working memory have been discussed by many researchers to have relationship with mathematics learning also. Poor digit span is one of the clinical characteristics, listed by Denckla (1979), for "pure" developmental dyslexia. Similarly, poor digit span and poor short-term memory is also implicated in number and calculation difficulties.

Although the primary problem of the dyslexic child is in language, reading, and spelling but, the child sometimes, manifests problems in arithmetic that are basically of a secondary nature. That is, the child is being provided extra instruction in language areas, sometimes at the cost of limited experiences in arithmetic, because he is taken out of the classroom sometimes during the time allotted to mathematics instruction. Other times child's reading problem is so overwhelming that hardly any attention is paid to other subject areas including arithmetic and mathematics. Therefore, as a result of the lack of experiences in arithmetic, and for no other apparent reason, the child may do poorly in arithmetic. Consequently, once the child's primary problem is reading and other language areas is remedied, the child may begin to do well in arithmetic, unless, by now, the failure in arithmetic has affected child's self-esteem. In that case, we will have to work on improving child's self-esteem first so that he may begin to do better in arithmetic.

Miles (1986) noted certain characteristics in dyslexic that included some specific problems in learning arithmetic.

> "[N]oting difficulties and errors which seemed incongruous, weird looking misspellings: for example, confusion of b and d, uncertainty over time and dates, and inability to learn arithmetical tables.
>
> "In particular there is reason to think that dyslexic children do not simply grow out of their difficulties as they become older. Thus both Critchley (1970, p. 114) and Rawson (1978, p. 91) report slowness at reading in dyslexic adults, while Miles (1984) reports that dyslexic students at college or university were slower than their peers at a variety of tasks involving the processing of symbolic information" (Miles 1986, p. 102-106).
>
> "There is further evidence of anomaly in the fact that, when symbolic material has to be processed at speed, dyslexics have been found to be slower not merely than age-matched controls, which would not be unexpected, but than controls matched for reading age' (Baddeley et al, 1982) or spelling age (Ellis & Miles, 1977)" (Miles, 1988, p. 97).

7. Mathematics Learning Problems due to Visual Dyslexia

The form and aspect of dyslexia that affects arithmetic commonly and interferes a great deal with mathematics learning is that of **visual dyslexia.** This is basically the inability to translate printed language symbols including mathematics symbols, into meaning.

> "The underlying flaw in visual dyslexia is the child's inability to comprehend order or sequence. Generally, this disability functions whenever the dyslexic tries to deal with time, arrangement, or relationships of entities along a continuum" (Jordan 1972, p. 13).

Most visual dyslexics see certain letters, numbers, and symbols backward and upside down. To read whole words in the context of a sentence is a jumbled process for such a child. Not only does he perceive individual letters incorrectly, but he also sees part of words in reverse. Because of this sort of scrambled perception, visual dyslexics are forced to work very slowly. Their arithmetic work, as a result, is very slow and deliberate. They may not finish assignments and tests in the allotted time.

Visual dyslexics are also generally handicapped in any situation which requires them to comprehend sequence; therefore, this child's major handicap is the inability to visualize printed symbols in correct sequence or position. This may disturb work in the areas such as place value and reading large numbers. They have difficulty in carrying out sequential instructions — an essential element in learning elementary mathematics and in the execution of procedural mathematics.

Denckla and her associates found that dyslexics were poorer than mental age matched controls on tests of object naming and slower on rapid automatized naming (Denckla & Rudel, 1976a, 1976b). Naming problems is part and parcel of their disability (Snowling, 1987). The reading disabled children are less accurate in labelling the objects and have particular difficulty with low frequency and polysyllabic words (Katz, 1986). This aspect of disability has a carry over effect on mathematics learning. This applies particularly to recalling and naming geometric shapes and describing terms in arithmetic and almost all aspects of mathematics.

8. Mathematical Problems due to Dysgraphia

Another related problem of some dyslexic children that may interfere in learning mathematics is **dysgraphia,** the inability to cope with handwriting. Dysgraphia involves faulty control of the muscle systems used to encode letters and numbers accurately. But even when there is no motor coordinator difficulties, faulty visual perception can cause writing difficulties. When an undistorted image is picked up by the eye and sent to the seeing portion of brain and is there distorted through improper visual perception, this distorted image will then be reproduced in this improper fashion by the hand. Or, if a distorted visual image is stored in a visual memory area, the hand will reproduce it later in its perceptually distorted form, even though the brain's motor mechanism for fine coordination is functioning properly. The dysgraphic student usually has a clear mental image of what he intends to encode, but he finds himself "forgetting" how to write specific symbols. One of the effects of this type of dyslexia is the inability to perceive parts of a whole in relation to the whole. Misunderstanding of place-value concept is an example of this difficulty.

> "An earmark of dysgraphia is the inability to copy simple geometric shapes without distortion. A frequently observed flaw in copying shapes is the tendency to draw 'ears' at the corners of simple figures" (Jordan 1972, p. 61).

Children with dysgraphia problems (writing, drawing, etc.) have problems in learning the motor pattern for writing either letters or numbers, making drawings and illustrations for mathematical concepts.

Children with dysgraphia, therefore, may have varying levels of difficulty in written aspects of arithmetic and mathematics: confusing directionality; written numerals poorly; distorting simple geometric shapes; failing to close shapes; having difficulty reproducing simple designs from memory; determination of procedural and written work; deteriorating towards the end of a mathematics exercise; writing one thing and reading it as something else; having difficulty staying on lines and aligning numbers and procedures.

Dysgraphics are extremely awkward in any task involving manipulation of small objects e.g., miscounting objects in early number learning. This creates difficulties in early arithmetic instruction. They have difficulty grouping objects by shape, size, color, texture, or other physical characteristics. All of these difficulties affect their mathematics work. Also during the elementary and secondary school years dysgraphia may cause another difficulty, for example,

> [A]n examination or a test involving mathematics could be quite an ordeal for those dyslexic children who experience the difficulties which we have described. One boy reported that he had wasted about three-quarter of an hour in an A level examination* through misreading the direction sign (<) on a vector line...He had realized that his answer must be wrong but could not understand why (Miles & Miles 1975, p. 58).

Here is another example:

> I took the Cambridge Entrance Examination a year before my A levels.* The questions were very varied and it was important to read them carefully. We tried to arrange for these examinations to be read to me, but that request was turned down. However, the first examination drained me to the extent that I was almost incapable of reading or writing at all, so the rest of the papers were read to me...I successfully got a place at King's College, Cambridge to read maths, in spite of having no language qualifications at all. I went on to do A levels in pure maths, applied maths, and physics, but more reading and writing was involved than 0 level" which made the examinations much more of a struggle (Jansons 1988, p. 500).

Many of the problems that dyslexics face in arithmetic can be put right with proper teaching; and in particular the use of concrete instructional materials "may help to generate the appropriate lexical entries. Thus, just as a multisensory approach helps in the building up of letters, combination of letters, and words, so the visible and tangible presence of say, seven objects, the mark on paper, '7', the written letters 's-e-v-e-n' and the sound 'seven' can jointly contribute to the formation of the same lexical entry" (Miles 1983).

An important theme running through the more recent research work on dyslexia in children is that the nature of their deficit changes with development. This has implications to their ability to learn mathematics and our planning of remediation programs for them. It is essential that we adopt a developmental framework to help us understrand the effect of reading failure on mathematics learning.

Of course, there are many differences based on strength and weaknesses of dyslexics and dyscalculics. For example, Ellis (1981) has proposed that dyslexics have problems when visual stimuli have to be named. Not all dyslexics have problems in mathematics and similarly not all dyscalculics have difficulty in reading and other language skills. We have observed that about forty percent of dyslexics have difficulty in learning basic arithmetic and mathematics.

Sometimes, some of the carryover problems can be contributed to by poor teaching and lack of appropriate mathematics experiences also. This kind of problem is really not a learning disability, in the true sense, but a learning difficulty in mathematics. As soon as a student encounters a good teacher his difficulty begins to disappear.

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*A level examinations are University-entrance examinations in the United Kingdom."

Not every child who has reading and language disability exhibits mathematical learning disability. Of course, many students with reading and language disability may possess characteristics that make them above rather than below the average in mathematical skills. It is true, however, that the difficulty in reading frequently complicates the situation in such a way as to make the individual appear deficient in mathematics. Sometimes the nature of mathematics itself is a help for a dyslexic student, some of them may possess very good visualization abilities, and, therefore, can learn mathematics easier, as the following example clearly demonstrates:

> From an early age I found that many things were easier to think about without language. This usually, but not always, meant thinking in terms of pictures and was particularly true when trying to make or understand intricate mechanisms. It is also the way my father thinks about things. I soon found it reasonably easy to create images in my mind and to manipulate them and, although I do not have a photographic memory, I developed a very good visual memory. For example, in building Meccano models (an engineering based construction toy) I would try out many possible arrangements in my mind before any assembling of the pieces.
>
> For my first year at Millfield, I was in a very low mathematics class and I was not studying physics. One of the teachers seemed to appreciate that I was bored with maths***, not because I could not do it but because it was not stretching enough, so I was moved up to one of the higher maths classes taught by the head of department, Mr. Sherlock. He was an extremely good teacher and soon I became much more interested in maths than I was in either biology or chemistry. The very compact notation, which I had no trouble in reading or writing, excited me and allowed me to play around with mathematical ideas for a long time after reading only very small amount (Jansons 1988, p. 400).

Sometimes the emphasis is on reading and other language related remediation, as a result, a student may not get enough experiences in arithmetic and therefore may exhibit difficulty in this area as the following clearly illustrates:

> My mother became quite anxious as the time approached for me to take the 11 plus examination ... I failed the examination quite dramatically as I was unable to read a few of the questions and was then almost unable to write even the shortest answers. There were a few arithmetic questions which I could do although I lacked practice as I had been taught with very slow children (Jansons 1988, p. 499).

Remedial Strategies:

The above examples clearly show that unless special compensatory arrangements are made, the unrealized potential of a student in mathematics may never be realized because of the reading and language related problems that come in the way of a student while he is learning mathematics. Because of support and understanding of special situations on the part teachers, Mr. Jansons went on to a Ph.D in mathematics at Cambridge University.

In cases in which the child is experiencing difficulty in reading as well as in mathematics, remediation must include work in reading as well as in mathematics. We have observed, that as the child's reading skill increases, he is able to use whatever mathematical knowledge he has and is frequently found to have much better understanding of the subject than was evident before the reading disability was overcome. However, in some cases, because of the delayed remediation, there may always be some gaps in number comprehension. It is important to fill in these gaps as early and as rapidly as possible rather than to drag the child through the review of things that the child already knows.

The key element in the planning for remediation for the children exhibiting the carryover type of problems is to attend to both components of the problem: the mathematical problem and the problems that are contributing to the problem in mathematics. If the focus is on the mathematics problem alone the success will not be long lasting as the cause of the problem is not taken care of. It may still exist. For example, in cases in which the child is experiencing difficulty in reading as well as in mathematics, remediation must include work in reading as well as in mathematics. We have observed, that as the child's reading skill increases, he is able to use whatever mathematical knowledge he has and is frequently found to have much better understanding of the subject than was evident before the reading disability was overcome. However, in some cases, because of the delayed remediation, there may always be some gaps in number comprehension. It is important to fill in these gaps as early and as rapidly as possible rather than to drag the child through the review of things that the child already knows.

B. **Developmental Mathematics Learning Disabilities**

The second category includes those problems in mathematics which are indicative of the child's lack of conceptualization and lack of mathematical thinking indicating lag in quantitative thinking or failure in mathematics is due to lack of adequate number concepts. Problems of this type are seen, at one level, when the child is showing difficulty in gaining competence in mathematics due to his lag in cognition necessary for mathematical concepts. In other words, the concept which he is faced with is not at his "natural" level of thinking and cognitive preparation. The difficulty emerging from this kind of situation can be termed as lag in quantitative thinking.

Each arithmetic operation is an integration of various complex cognitive and behavioral skills. To calculate effectively one must be able to (a) process the numerical information, that is, perceive, comprehend, and produce numbers; (b) process the operational sign that indicates the specific calculation to be performed; (c) access arithmetic facts and information, i.e., 2 ×7 = 14; 6 + 4 = 10; 10% = .10; and (d) execute the retrieved calculation procedure(s). All of this processing calls for the interaction of various

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**'O' level examinations are Secondary School learning examinations in the United Kingdom.

***A term used for mathematics (particularly in Commonwealth countries).

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prerequisite skills for learning such as: sequencing, pattern recognition, spatial organization, working memory, etc.

Children of normal intelligence and with no apparent reading disability may fail in arithmetic because of two main types of difficulties: (a) lack of sufficient skill in fundamentals to enable them to work rapidly and accurately even when methods of solving the problem are correct and (b) inability to conceptualize and execute the solution process in solving a problem.

a. In the first situation the child makes errors in addition, subtraction, multiplication, and division facts with the result that his answers are wrong no matter how correct his methods of attacking the problem may be. The work these children have done frequently consists of pages of problems in which the answers are wrong although the methods of solution (the procedures) are correct.

b. In the second situation the child attempts to use formal results of rote learning to solve problems without having any concept of what types of facts are involved. This particular tendency is characteristic of intellectual deficiency but it may be found in the cases of individuals of normal and superior intelligence also who either lack number experience or who have been taught number (arithmetic) and related mathematics in such a rote way as to make it meaningless for them.

Some of the obvious examples of these types of problems are:

★ Lack of estimation skills: the child cannot (meaningfully) determine how many articles are in a group, cannot determine (estimate) the possible results of simple arithmetical operations. For example when a child is asked to find the product of 12 and 14 and his estimate is a number as low as 50. This shows that he has not understood and conceptualized the meaning of multiplication.

★ Inability to understand and record process signs correctly and difficulty in associating between the symbol meaning and the quantity, between visual and auditory symbols i.e., says one thing and writes something else.

★ Inability to establish one-to-one correspondence between two sets of objects and differentiate between ordinality (order) and cardinality (quantity) of a given set.

★ Inability to grasp the conservation of number, length, area, volume and weight, i.e., a different arrangement does not change the quantity, length, area, volume and weight when their configuration is changed.

★ Inability to choose a method or operation to solve a given problem and then the inability to follow and remember the sequence of steps used in various mathematical operations and algorithms.

★ Inability to determine the component primary concepts in complex and secondary concepts.

★ Difficulty in symbolizing and communicating a particular type of experience. Problems in generalizing experiences, drawing conclusions, or ideas from experience that would ordinarily lead to understanding of number, quantity and mathematical concepts. Some students have difficulty generalizing their skills from one classroom to another, let alone applying their knowledge to solve real-life problems.

★ Another type of difficulty with generalization consists of failure to apply previously mastered skills to solve related problems. For example, a student may correctly solve problems of the type [   ] + 4 = 9, but be unable to solve more complex problems, such as [   ] + 40 = 90. Failure to generalize from one setting to another or from one problem type to another can be due to a number of factors, including: (a) inability to perform all components of the behavior, (b) failure to recognize the similarity between stimuli, and (c) insufficient reinforcement of correct responses.

★ Problems in integrating the experiences from different modalities: auditory, visual, and tactile information.

1. Dyscalculia and Lowered Achievement in Mathematics

Numerical calculation is a complex process involving the interaction of various cognitive mechanisms. Processing information (cognitive functioning) involves more than reception and expression. Information processing involves analysis, synthesis, decision making, and a host of other executive functions. Learning disabled children do not process information effectively. They omit information, include unnecessary information, forget steps, and include the wrong steps. Difficulties of both types (mentioned above) are due to lack of adequate number conceptualization and integration of these cognitive mechanisms.

A disruption to any part of this complex system will result in a calculation impairment. The particular form of impairment is determiend by the locus and nature of damage to the cognitive system. The damage, impairment, or the absence of skills can be manifested in one or all of the following:

1. difficulties in the reading and writing of numbers,
2. difficulties in the spatial arrangement of numbers,
3. difficulties in the execution of arithmetic operations.
4. difficulties in retrieval of facts and procedures appropriate to a particular problem situation.

a. Dyscalculia vs Acalculia

Individuals having difficulties in learning mathematics manifest the above symptoms in varying degrees and forms. One of these form is known as dyscalculia. Which has several types including developmental dyscalculia, acalculia and anarithmetia. Terms such as acalculia and dyscalculia have been used with definite and specific meanings by individual researchers. At the same time there is no definite agreement on their use universally in the literature relating to learning problems in arithmetic and mathematics. They have not been used consistently by various researchers and authors describing calculation disorders. Although, there are significant differences in the conditions of acalculia and dyscalculia, some authors have used the terms interchangeably. For example, the terms, developmental or acquired dyscalculia and acalculia, are used to describe the general inability to calculate, use numbers and number concepts, or simply the inability to perform the operations of arithmetic. In the literature on learning problems in mathematics, the descriptions of these terms are quite diverse, to say the least.

We will use these terms as they have appeared in the literature related to learning problems in mathematics. Accordingly, **dyscalculia** refers to a disorder in the ability to do or to learn mathematics, i.e., difficulty in number conceptualization, understanding number relationships, and difficulty in learning algorithms and applying them. It is characterized by substantially lowered arithmetic achievement, i.e., several years below the appropriate level. Sometimes, their level of functioning is as much as two to three years below the expected chronological age level achievement. It is much more than a matter of forgetting a few rules or procedures once in a while. The dyscalculic individual may have sufficient intellectual ability and proper motivation, yet, the individual will show a lower than average mathematical age in relation to normal mental age. Dyscalculia, by some researchers, has been defined as the difficulty in mastering mathematical concepts and/or computations and inability to perform simple arithmetic operations. For example, "developmental arithmetical retardation in children [dyscalculia], a mild or moderate acalculia, means faulty calculation ability (Gaddes, 1985, p. 355).

Developmental dyscalculia is considered "as a disorder of the abilities for dealing with numbers and calculating which is present at an early age and is not accompanied by a concurrent disorder of general mental functions;" in contrast to 'acquired dyscalculia' where there must be an indication that the child's arithmetical abilities were formerly at a higher level and have subsequently deteriorated."

Kosc (1974) defined dyscalculia as: "A structural disorder of mathematical ability, which has its origins in a genetic or congenital disorder of those parts of the brain, that are the direct anatomic-physiological substrates of the maturation of mathematical abilities adequate to age without simultaneous disorders of general mental functions."

Acalculia also refers to calculation and number processing deficits, but it is the loss of fundamental processes of quantity and magnitude estimation (Guttman, 1936). Therefore, acalculia, according to some authors, is an acquired disorder of the power of calculation in patients who hitherto have shown no disturbance of their arithmetical faculties. They have acquired this complete or partial inability to deal with numbers due to a focal lesion which might have been caused by some insult or injury to the brain (Gaddes, 1985, p. 355).

Acalculia can be an adult disorder that is acquired after mathematical ability was once intact. Dyscalculia, on the other hand, may be considered as a developmental disorder in which mathematical ability has not been and cannot easily be acquired. In short, acalculia and dyscalculia can both refer to a general mathematical disability.

Acalculia, to some other authors (Kosc, 1981), is the loss of the entire (or a large) system of calculation, but dyscalculia is the loss of a part of the system which may lead to impairment of the entire system. These authors used the terms acquired and developmental to differentiate adult from childhood deficiencies.

b. **History**

Although the mathematics disabilities of the dyscalculics have been observed and described for about one hundred years, we still know very little about dyscalculia and acalculia and their definite causative factors. Most of what we can say about mathematical difficulties relating to dyscalculia comes from the study of brain-damaged adults. Neurologists first observed dyscalculia in brain injured adults who were unable to perform calculations after injuries. The case studies of adult patients who had sustained brain damage and others show that the calculation disorders, characterized by a consistent error tendency in some portion of the computational process, are extremely common in patients referred for neurological consultation, even though acalculia [or dyscalculia] may not be part of their presenting complaint (Spiers, 1987).

Henschen (1920) was the first to use the term acalculia in describing three types of mathematical disorders in case histories in literature. He saw no relationship between arithmetic reasoning and intelligence or language and musical ability. Hans Berger (1926) differentiated between **primary acalculia,** a serious interruption of ability due to lack of calculational reasoning, and **secondary acalculia,** interruption due to a processing deficit; such as memory, language, or attention dysfunction. Kleist (1934) noted dysfunction which was due to constructional **apraxia,** the lack of integrating parts into wholes.

The inability to acquire mathematical skills has also been studied by psychologists and special educators. But, in contrast to developmental dyslexia, developmental dyscalculia has been little studied. The paucity of research work in this field contrasts with the enormous literature on developmental reading problems. Only very recently have educators begun to study this condition and they are now beginning to develop and implement the remedial procedures for these students in clinical settings.

The nature of the mechanisms of arithmetical computations and the representation of numbers in the normal adult brain has received new stimulus from recent studies of acquired dyscalculia.

The largest number of patients with acquired dyscalculia have been studied by Hecaen (Hecaen & Angelergues, 1961; Hecaen, Angelergues, & Houllier, 1961). Patients were classified into three groups: alexia or agraphia for digits and numbers, in which the reading and writing impairment affect arithmetical operations; spatial dyscalculia where there is a disorder of spatial organization for numbers leading to lack of respect for the rules used to place digits in their proper order; and anarithmetia in which neither of these deficits are apparent but there is difficulty in performing arithmetical operations.

This tripartite classification assumes that normal number processing and calculation involves a number reading and writing ability, a spatial ability, and calculation ability, but the nature of these posited abilities is left unspecified. It is not clear how each ability is involved in different tasks, nor the sort of impairment which should result from its disruption. It is also a problem that the three types are insufficient to account for the great diversity of number processing and calculation impairments seen in brain-damaged patients.

c. Neurological Factors: Functional vs Lesions in the Brain

Traditional attempts to classify acquired dyscalculia into different types had been based largely upon two variables: the functional factors causing the disorder and the lesion locations associated with the disorder (Boller & Grafman, 1983; Temple, 1989).

The functional factors proposed include disorders of: attention, memory, intelligence, spatial ability, language, abstraction, and body schema. However, some scientists (Temple, 1989), do not think that it is "clear that any of these potential concomitant deficiencies are necessary and/or sufficient conditions for the emergence of a calculation disorder. Looking from the lesion location perspective frontal, occipital, temporal, and pareital areas on both sides have been implicated" (Temple, 1989).

Goldstein (1921) investigated the behaviors of brain-injured soldiers and compared these records to the postmortem information. Mathematical dysfunction was mentioned in his writing. Sittig (1921) noticed calculational difficulty due to agnosia for symbols, such as +, −, × and ÷. **Agnosia** means inability to read and understand the meaning of the symbols.

Luria suggests that "different parts of the brain are activated by the use of different verbal association networks. Changing a word may change a whole network of brain connections" (Farnham-Diggory, 1978). As a result, the arithmetic disorders can be thought of as a manifestation of specific difficulties in specific parts of the brain.

It is believed, as a result of electroencephalograph (EEG) studies with older children and adults, that most arithmetical activity is specialized in the left hemisphere. "Therefore, perhaps, most dyscalculic children are having more difficulty with math than are their peers because they persist longer in their tendency to rely on holistic cognitive strategies to deal with arithmetic problems that are best solved analytically." This approach "links dyscalculia to cognitive style rather than to deficit … Dyscalculic children may not realize that the analytical operations they possess are relevant to the task at hand, i.e., mathematics. (Weinstein, 1980)

d. Neurological vs Cognitive Factors

Gerstmann (1924, 1927, 1930, 1940) discovered a series of symptoms that he believed were indicative of the brain lesions in the left parietal lobe of the brain. These symptoms were agraphia, acalculia, right-left disorientation, and finger agnosia. Gerstmann wrote that these disorders all involved the digits in some manner. His writings, in 1924, marked the beginning of a controversy that clouded investigation into mathematical disorders by focussing attention on whether the syndrome existed, rather than finding out why some people cannot function in mathematics even when they can succeed in language skills.

Ginsberg (1977), a psychologist, has studied children's natural strategies for arithmetical operations in his book, **Children's Arithmetic,** and he identifies and demonstrates several types of error patterns in children's mathematical behavior. Most of these patterns are analyzed in terms of the cognitive complexity of the task involved and cognitive preparation of the child. One of the ways to evaluate the child's cognitive preparation is to see his performance on Piagetian tasks.

Rourke (1982) has looked at the calculation system in developmental dyscalculics. He compared developmental dyscalculics who were good readers and those who were poor readers. The good readers misread signs, aligned rows and columns inappropriately, and missed entire calculation steps. The poor readers avoided unfamiliar operations and had problems with tables, and in recalling appropriate calculation procedures.

A combination of Piagetian tasks and Gerstmann syndrome was studied in two 9-year-old children with numerical difficulties (Saxe & Shoheen, 1981); IQs of 100 and 81 were reported for the two youngsters. The tasks studied were as follows:

Gerstmann Syndrome:

- ★ Left-right orientation
- ★ Finger agnosia
- ★ Graphic Skills
- ★ Mathematics behavior
    - rote knowledge of number system
    - applying the rote system
        - counting accuracy
        - counting to compare arrays: drawing task
        - counting to reproduce arrays
    - numerical calculations
        - single digit additions and subtractions

Piagetian Tasks:

- ★ Concrete operational taks
    - number conservation
    - class inclusion

45

- seriation
- liquid/quantity conservation

Both youngsters showed persistent deficits in all Gerstmann tasks. Performance on rote number tasks was deficient. For example, in writing numbers, B wrote 63 as 36 as he vocalized, "three, six."

Neither youngster was able to count fluently or meaningfully, nor was either able to use counting to demonstrate number. Both showed an awareness of the arithmetic symbols, but they could not do the addition or subtraction tasks.

Performance on the Piagetian tasks was also limited. Both children believed that changing the configuration changed the amount in the conservation tasks. Neither could arrange items in order. Both failed the class inclusion items.

Other researchers have looked for the cause of mathematics difficulty in some ways in similar manner. But rather than a defective brain as the cause for the problem they see it more as a developmental problem as seen from the generalized perspective of hemispheric specialization. For example, Weinstein (1978, 1980) believes that "developmental dyscalculia results from slower development, rather than impairment, of the cognitive structures underlying arithmetic. A neuropsychological explanation for developmental dyscalculia links the acquisition of number concepts to the development of 'hemisphericity' (i.e, differential activation of one or the other cerebral hemisphere)." This approach believes that "there may be a shift, as one gets older, from a tendency to rely on right hemisphere cognitive functioning (global thinking) to left cerebral hemisphere (analytical) strategies for problem solving."

e. Language Factor

Dyscalculia, in many ways, closely parallels the behaviors exhibited with language dysfunctions, therefore many neurologists believe that dyscalculia does not exist as a separate dysfunction, but is a manifestation of a brain lesion which is causing langauge dysfunction simultaneously. As a result, at the time that mathematical dysfunction was first discovered, many believed that these problems were extensions of the inability to read. Early work in understanding calculation difficulties related this difficulty to alexia which is loss of ability to read, and agraphia which is loss of ability to write. As early as 1917, Poppelreuter stated that difficulties in calculation always occurred with aphasia, the loss of ability to use or understand speech.

McCloskey et al. (1985, 1986) draw a basic distinction between a number processing system and calculation system. The number processing system comprises the mechanism for perceiving, comprehending, and producing numbers. Certain patients exhibit difficulty processing Arabic numbers (e.g. 6) whilst verbal number processing is intact (e.g. six). Other patients show the reverse dissociation. Thus different cognitive mechanisms are involved in the processing of Arabic and verbal numbers. Certain patients show intact comprehension on numbers by making appropriate magnitude judgments yet read the same item aloud incorrectly. Separate production and comprehension mechanisms are therefore postulated. A further distinction is drawn (within the verbal and Arabic number systems) between lexical and syntactic processing components. The lexical processing of individual number words and digits. Patients with impairments of this system make errors of the sort 4501 read as "four thousand and fifty-one." The syntactic component processes the relationship among the elements that comprise a number. Patients with impairments of this system make errors of the sort 4051 read as "four thousand five hundred and one" (Temple, 1989).

In contrast to number processing system the calculation system processes operational signs which indicate that a specific calculation is to be performed; it stores and assesses arithmetical facts; and executes retrieval procedures (Temple, 1989). The calculation system consists of the facts and procedures required specifically for carrying out calculations.

According to Denckla (1978), children with Gerstmann's syndrome "speak early and well, read well, do not have behavioral, attentional, or general athletic coordination deficiencies. They have difficulty with arithmetic and spelling, typically getting both numerals and letters out of order. They also have finger agnosia (trouble with names or other organizational schemes for ordering the fingers) and right-left disorientation. In arithmetic, they have particular difficulty utilizing place value of numerals and confuse the columns (ones, tens, hundreds, and so forth) as they carry out arithmetic operations ... their copying of designs and their map skills are often poor."

There are two schools of thought about Gerstmann's syndrome. Some (like Kinsbourne) view the underlying problem as one of sequencing, whereas others (Denckla) see it as a spatial-sequential mapping or matching problem.

Those who discount dyscalculia, as a separate problem than language problem, believe that the symptoms of dyscalculia, acalculia, agraphia, right-left disorientation, and finger agnosia do not exist alone, but always are part of a larger syndrome caused by a generalized lesion affecting ability in most areas of functioning. This controversy reflects the localization-lateralization theory. If specific areas of the brain are responsible for certain functions and if damage occurs in that same area in different patients, the same behavior dysfunctions should occur. If mathematical function is merely a result of language functions, there will be no specific area that controls arithmetic functions.

Several recent researchers have argued that the syndrome of dyscalculia should be considered separately from the language related problems and different from more general category of learning disabilities and even from learning disabilities in mathematics. In other words, dyscalculia and acalculia can exist independently of all other learning problems. That is, number of individuals may manifest dyscalculia and no other learning disabilities. On the other hand, some others may exhibit difficulty in mathematics and even dyscalculia or acalculia in the presence of dyslexia. By some estimate, the number of dyslexics who may experience arithmetic and mathematics difficulties (even dyscalculia or acalculia) is as high as 40 percent and vice-a-versa. It may appear, therefore, that the conditions of dyslexia and dyscalculia are both independent and dependent of each other.

f. Types of Dyscalculia and Acalculia

Kosc described six different types of dyscalculias whose combined systems are: inability to name, read, or write mathematical symbols and terms, operate the mathematical functions, understand the concepts, or manipulate real or representational objects (1981, p. 25-26). True acalculic means the inability to reason about mathematics. This form of disability seems to be rare, although the research into the incidence of any of the mathematical disabilities is limited. Gaddes (1985) correlated developmental dyscalculia with deficits in language (auditory-gnostic), reading (visual-gnostic), and/or spatial imagery and suggested that the mathematics problems in that context can be understood within the Piagetian model of development (p. 368).

The overall pattern of numerical neurological (Gerstmann), and developmental deficits suggests that arithmetical difficulties and cognitive limitations may not be isolated phenomena. Thus there is heterogeneity in each category. That is none of the skills or complex of skills identified as factors is responsible for the diverse calculation difficulties. They are neither necessary or sufficient. Therefore, mathematics learning problems in general and even the types of problems exhibited by dyscalculics, acalculics, and or dyslexics are not homogeneous in nature.

g. Manifest Behaviors of Dyscalculia and Acalculia

According to many authors the term dyscalculia meant general calculational dysfunction. These authors have identified specific types of calculation disorders. Various symptoms and characteristics have been developed. They include among others:

> *Paraphasic substitutions* — substitutions of one number for another number (e.g., 4 is read as eight, or used in calculation as a 3, such as 3 × 4 = 9); these substitutions have very little or nothing to do with the numbers involved in the problems;
>
> *Perseveration* errors are repetitions of the same number without the ability to move to the next intended digit (e.g., written stuttering 66666; adding 16 × 3 = 17 the next counting number; or repeating one of the numbers of the problem 6 × 3 = 3); Perseveration of operation refers to when a child has acquired a new operation or procedure and he applies that operation to places when it may not be appropriate (after learning "borrowing" in subtraction problems the child applies "borrowing" even when it is not appropriate or when the child has acquired operation of multiplication and now if you give him an addition problem he applies the multiplication procedure to addition also. For example,
>
> $$\begin{array}{r} 24 \\ \times 2 \\ \hline 48 \end{array} \qquad \begin{array}{r} 24 \\ +\ 2 \\ \hline 46 \end{array},$$ in this case + is treated just as multiplication.)
>
> *Reversals* of digits are digits fliped (e.g., 6 as 9; 4 as ㄣ); Reversals can also be numbers of two or more digits that are read inverted or misordered (524 as 425, or 245);
>
> *Slowness of responses* refers to a correct answer, but actual measured time of response is more than expected in normal situations. For example, giving an addition or multiplication fact in more than three seconds;
>
> *Misalignment* refers to numbers that are not placed in normal order during written calculations, as a result, the operation is performed inaccurately (examples: Find the sum of 6.24, .345, 23, 3.985 and .87. The child writes as:
>
> $$\begin{array}{r} 6.24 \\ .345 \\ 23 \\ 3.985 \\ .87 \\ \hline \end{array} \qquad \text{or} \qquad \begin{array}{r} 14 \\ +\ 12 \\ \hline 28 \\ 14 \\ \hline 42 \end{array}$$

Here the operations are performed incorrectly because of misplacement of operations and numbers.)

Luria (1966a) identified and studied four types of arithmetic disorders: **defects of logic; defects of planning; perseveration of procedures that are no longer appropriate;** and **inability to perform simple calculations.**

Under **logical defects,** Luria includes difficulties in understanding phrases like "a triangle below a cross" or "the brother's father." If such a phrase were dictated, the individual might write the elements in the order they were named, without regard to relationships having a spatial aspect. To grasp them, one must hold the elements in mind simultaneously and compare them in some dimension. Logical defects can also appear in the handling of numbers. For example, an individual might write the number 1029 as 129, thus demonstrating a failure to grasp the logic of zero as a place holder. Or the individual might write the number as 1000 29-in the order the numbers were named, without regard to the relationships signified. Some examples of logical defects, according to Luria have their origin in spatial problems. The loss of the ability to maintain reference points, such as the location of the tens, the ones, and so forth affects the person's understanding of numerical relationships. The person needs to hold several categories in his mind and interrelate them, this requires spatial and visualization abilities and when these "spatial syntheses are disturbed and ... spatial ideas have disintegrated, the categorical structure of number is fundamentally upset." (Luria, 1966b) Therefore, many tasks, such as understanding calendars and reading clocks, that are dependent on this integration, are also affected.

Defects in planning are manifested when an individual does not apply the preliminary analysis of the problem before he solves. These students "regularly fail to perform a preliminary analysis of the conditions of the problem. As a result, the patient was never able to formulate a plan for solving the problem. Instead, the patient jumped into impulsive arithmetic operations and lost connection with the original problem." Luria also says that these patients fail to verify their answers. Whatever answer they get they are "willing to settle for it."

Luria frequently studied patient behavior while solving arithmetical problems and asked them to verbalize it. He found that many times they continued to use the procedures that were no longer appropriate. If in the beginning of the solution process of a problem, the patient found something relevant to the solution process, he continued the same line of thought even after it became irrelevant or secondary to the solution process. This kind of perseveration made it difficult for patient to think or accommodate a new line of inquiry. The persevered thought process or a particular persevered approach continued to manifest itself in related problems and sometimes it continued even in the case of unrelated problems.

Inability to perform simple calculations: In this particular situation a person understands the logic of the arithmetic operations but cannot recall the facts automatically. According to Miles (1974) inability to recall arithmetic facts at an automatized level is one of the major characteristics of dyslexics. Which we believe they share with dyscalculics. Dyslexics cannot rapidly access verbal labels and arithmetic facts as they have problems in retrieval from long-term memory (Spring & Capps, 1976; Baur & Ernhert, 1984). Similarly,

> "...difficulties which dyslexics experience are associated with the lexical (name) coding of both visual and verbal information. Thus, dyslexics have problems with short-term memory function and with the retrieval of stored memory traces (Snowling, 1987, p. 4)."

Since these children cannot recall facts automatically, they generally derive these facts by sequential counting. The finger counting for deriving results is quite prevalent in these children. What exactly is the nature of this defect? It is not simply a loss of number memory because the student is able to retrieve any number by counting up to it. But counting seems to be the only number-manipulation scheme available to them. The students do not lose the numbers; they have lost the schemes to put them in or the patterns of their relationships.

h. **Identification/Screening of Dyscalculia/Acalculia**

Although dyscalculia has been mentioned in literature for more than 100 years and has been studied by researchers for more than three decades, the procedures and methods of diagnosing and evaluating the disability have not been consistent with each other nor have they been complete. The other related problem of evaluation is to effectively diagnose the condition and the factors responsible for its incidence. For example, even when standardized methods of evaluation were used, the measures did not evaluate errors that are symptomatic of dyscalculia. There are hardly any norm referenced, standardized methods to evaluate the condition of dyscalculia that can differentiate and discriminate between the calculational errors exhibited by neurologically impaired, learning disabled, and normal students. Such type of discrimination is necessary from several perspectives: first from the point of view of etiology of the problem; second, the nature of the remediation to a great extent is dependent on the nature and cause of the problem; third, one needs to know the extent of resources and personnel needed to deal with the problem; and finally, from the perspective of the services available.

For example, most standardized tests are achievement tests. They test the acquired knowledge base of the child in that content area. They do not evaluate the cause for the symptoms of dyscalculia in particular and/or mathematics learning disability, in general. Therefore, there is a need to devise better diagnostic tests to clearly identify the problem and begin to understand the problem of acalculia and dyscalculia. We need also to distinguish between slow learners and possible dyscalculics. We need to identify:

First, individuals whose mathematics achievement is significantly below age level as predicted when their I.Q. and cognitive development is taken into account. That is, these individuals can be thought of as slow learners in arithmetic and/or mathematics since their arithmetic/mathematics ability as predicted by their I.Q. and cognitive functioning is poor but so are their attainments in other areas.

Second, group of poor achievers in mathematics are those individuals whose performance is significantly below the level expected given their age and intellectual/cognitive ability. For these individuals there is a significant difference between their predicted and their actual mathematics achievement. Some individuals in this group may be considered as dyscalculics if we can rule out the educational factors such as poor teaching and lack of preparation for mathematics experiences (frequent school absences, lack of motivation, emotional and social problems, inadequate opportunity for learning, etc.). The difference in achievement and expected level of performance should be at least two years before we could consider the possibility of assessment for dyscalculia.

i. **Remedial Strategies for Dyscalculics**

1. It would be important to employ a cognitive or discovery-oriented program of remediation with children whose difficulties are projected to be developmental and are of a long-term character. In this sense, the emphasis it no only to provide competence and skills in the area of mathematics but we should also provide experiences that will improve child's cognitive development and provide him with metacognitive skills in order for him to become an independent learner.

2. By contrast, the problems of some children are specific and may require a more behavioral or direct instruction program. These individuals may have extreme difficulty changing from one topic to another. They may experience difficulty in mastering a single operation. The explicitness of instruction and the amount of time devoted to the task may require a concerted effort by the teacher.

C. Failure in Mathematics in Cases of Mental Deficiency

Since even the simplest of arithmetic and mathematics processes require reasoning and one of the essential characteristics of the brain-injured (or mentally deficient) is inability to reason, it is obvious that these children and adults are unable to develop normal skills in this subject under normal teaching conditions. Most research shows that their competence in mathematics is severely limited. The disorders of these individuals are presumed to originate in some type of central nervous system disorders. Which may be due to insult or brain damage, those caused by inefficiencies in the integration of utilization of neural mechanisms (e.g., exchanges across the corpus callosum; Kinsbourne, 1983), and those due to developmental variations (e.g., maturation in hemispheric specialization). The reverse is also true: If the child can do arithmetic and mathematics with understanding that shows that there is no mental deficiency.

Some of the following characteristics may be responsible for their difficulty in learning and utilizing mathematics skills:

- ★ attention disturbances and attention span
- ★ figure-ground pathology
- ★ dissociation
- ★ memory dysfunction
- ★ sequencing difficulties
- ★ discriminating difficulties
- ★ difficulty with directionality and body parts orientation
- ★ spatial and temporal disorientation
- ★ obtaining closure
- ★ intersensory disorganization

Any remedial instruction for these students has to keep the above factors into consideration. Without proper teaching, many brain-injured individuals may learn to go through the mechanical operations of computation but they cannot use what they learn. The numbers such an individual says and writes are meaningless names or words to him and have little if any concept connected with them. There have been cases of mentally deficient individuals who knew all the simpler number combinations and yet not be able to solve the simplest problem making use of these combinations.

D. Mathematics Anxiety

From eighth grade on, the percentage of students, in American schools, taking mathematics decreased by half for each year of schooling. Mathematics is considered by almost all segments of the school and society as the worst curricular villain in driving students to failure in school. More than any other subject, mathematics filters students out of programs leading to scientific and professional careers. This decline means that each year, fewer and fewer sudents get mathematics and therefore science and technical courses that are necessary for important professional, scientific and technical careers. In many cases this decline in interest in mathematics is the result of mathematics anxiety.

Mathematics anxiety or mathophobia is now a well known phenomenon. This section deals with the nature of mathematics anxiety and its effect on mathematics learning.

Mathematics anxiety is the manifestation of a person's fear of mathematics. Math anxious person avoids contacts with mathematics and mathematics related situations. It can range from mild displeasure in dealing with quantitative information to complete avoidance of any thing that uses or resembles to mathematics. Its symptoms range form subtle uneasy feeling when one mentions mathematics to pathological signs of perspiration, sweaty palms, and overt uneasiness about anything quantitative. Some individuals with math anxiety may know the specific reasons for their anxiety. Others, on the other hand, have just a general uneasy feeling about mathematics.

Math Anxiety may be either expressed or concealed. Some talk about it constantly expressing how incompetent they feel in mathematics. Others go to great lengths to hide their incompetence. They invent unique methods of avoiding mathematics. Most math anxious people generally develop compensatory mechanisms to make up for their deficiency.

Here is an example of the behavior of a person with math anxiety:

> "A person who comes to his math teacher with a slip to drop a course in finite math and who gives the reason for dropping the course 'I am good in algebra but I don't like this probability stuff. I made a mistake. I should have taken algebra and trig instead of finite math' is perhaps experiencing the same math anxiety as a person who reacts in a hostile manner in a calculus class at the sight of a radical sign" (Datta & Scarfpin 1983, p. 1)

1. Nature of Math Anxiety

The nature of the math anxiety is defined according to the behaviors exhibited by the individual and the causes that are responsible for it. Consistent failures in mathematics result in mathematics anxiety. Failure in arithmetic can cause emotional problems among some children. When parents and teachers give irrational reasons for these failures, children may develop negative feelings towards mathematics. These feelings then give rise to mathematics anxiety which in turn may cause blocks in learning mathematics.

During early elementary school a child's mathematics anxiety is only a symptom of something not right. It indicates that a child is having difficulty in learning mathematics. It calls for an attention to the fact that the child needs help in learning mathematics. The origins of this learning difficulty may lie in several areas: cognitive immaturity, lack of preparation, lack of concrete models for mathematics concepts, high expectations, etc. But the fact is that they are creating conditions of failure for the child. If not attended to early enough the child begins to feel anxious about mathematics. These patterns of failure solidify mathematics anxiety. If the failures and the resultant anxiety continue into the later school years, the mathematics anxiety becomes a causative factor for the child's learning difficulties in mathematics. It is a causative factor, in the sense, that the student, because of math anxiety, avoids mathematics and mathematics related activity. This avoidance, in turn, creates a lag in his mathematical preparation and thinking. This lack of preparation for mathematics creates conditions of failure. The studnet then fails. This same pattern is repeated again and again reinforcing his math anxiety.

The condition of fear or avoidance of mathematics on the part of children, in the early school years, is generally overlooked by the child's teacher and in many cases even by the parents. At this point it may just be a symptom of immaturity in mathematics thinking or a symptom of mathematics difficulty. Their sincere hope is that the child will outgrow the fear and with maturity will learn to like mathematics. In many cases, that is exactly what happens. But this is not true in many other cases.

The other reason for mathematics anxiety is related to socio-cultural factors as this example illustrates:

> "Ms. Cook needed to take a math course to graduate. For some reason she put it off till her last semester at the University and she was taking an elementary course called 'Topics in Math,' designed especially for liberal arts majors and taught by one of the authors. But she was having problems — nothing made sense, she did not understand anything. She dreaded coming to class, in the class she always felt stupid. Then, several days before the first quiz, she started becoming so nervous that she could hardly sleep. She did not take the quiz, avoided the class and avoided the teacher. One day, the teacher happened to see her in the Unviersity cafeteria and asked her what was wrong, why she did not take the quiz. Rather hesitantly, she started talking and said 'Math makes me sick, I don't need math, I know math is not for girls.'
>
> "Then, with tears in her eyes, she told how her elementary school teacher used to tell her 'Go ask your daddy' anytime she had difficulty in arithmetic. Unfortunately, she did not have a daddy around because she was living with her mother only. Most of the time she never asked her mother to help her with arithmetic. One day she did ask and her mother got really angry at her and the teacher, so much so that she went and told Ms. Cook's teacher, 'Why are you doing this to her? I never learned this stuff in school. You know as well as I do that math is not for girls.' Since then Ms. Cook had been avoiding math with the belief that math was not for girls. Therefore, when she needed to take a math course to graduate, she experienced anxiety which she could not overcome" (Datta & Scarfpin, p. 2).

Math anxiety and the resultant distaste and avoidance of mathematics, in this case, is caused by socio-cultural factors. The common causes of this type of math anxiety are:

1. The female fears of competing on man's turf.
2. The myth that boys naturally do better in math than girls.
3. Distrust of intuition, especially as applied to math.
4. Common myths about mathematical ability.
5. Illusion of math as an exact science.
6. Individual experiences in school, at home or society in general.
7. Lack of information about mathematics learning processes.
8. Poor mathematics teaching.
9. Myth that creativity in the humanities and mathematics learning are mutually exclusive activities.

Any one or more of the above factors may cause a person to either avoid mathematics experiences, encounter failure in mathematics, have negative feelings about mathematics, and therefore, develop math anxiety.

Mathematics anxiety is not an indication of lack of intellectual functions always. Many a times a person with math anxiety may have all the cognitive abilities necessary for doing well in mathematics. But, on the first instance of their failure in mathematics they may resort to any of the above factors as their excuse for failing in mathematics. The excuses are then reinforced by the socio-cultural conditions. This is particularly true in the case of many young girls (of junior high and high school age groups) and women in the western society. Their impressions about mathematics are further reinforced by many teachers, school counselors, administrators, parents, and other significant persons in their lives. Particularly when these persons are not aware of how mathematics is learned and how the learning of mathematics can be improved. They perpetuate the myth of mathematics being difficult for girls.

Individuals exhibit two types of mathematics anxiety: **specific** and **global.**

a. Specific Math Anxiety

An individual who is performing well in every other academic area except mathematics, but has long-term unsuccessful experiences in specific aspects of mathematics; demonstrates anxiety about mathematics; mildly dislikes mathematics related experiences; and is willing to admit incompetence in specific aspects of mathematics suffers from the type of mathematics anxiety which we term **specific math anxiety.**

This individual does not dislike all mathematics but he is anxious about some specific area of mathematics. Generally, he has tried hard to understand that particular part of mathematics and now as a result of those unsuccessful and frustrating experiences has developed anxiety about mathematics. These are the people who get into a course and then find some excuse to drop the course. These students generally, complain about the teacher, about the textbook, and any thing else that might relate to the mathematics experience. As soon as that thing is changed they feel they will be able to learn mathematics.

This kind of anxiety is triggered by certain, specific math experiences. It may be a symbol, a certain procedure, a concept, or a mathematical term. For some reason that specific mathematics experience creates a mental block in the process of learning new mathematics concepts or review of something that the student might have been exposed before. Since that particular mathematics concept or procedure makes that person uncomfortable the person is not able to go further.

We have observed that the reaction of persons with the specific math anxiety is also very specific. When they come to the Center they have specific goals about mathematics and have specific need and their reactions about mathematics are also specific. For example, they are apt to say:

"My teacher has started doing geometry in class and I always have difficulty in mathematics, can you help me go over this part of the course"

"I have to take these exams, I always do poorly on exams, can you help me in passing these exams"

"I used to be good in math up to sixth grade"

"I like geometry but I get lost in algebra"

"I like arithmetic and algebra, but geometry is something else"

"I had a really bad math teacher in eighth grade, it was all downhill after that"

"I understand what you are saying, but I don't see the meaning, I am sorry"

"Calculus is so abstract"

"Why can't you explain the way my sixth grade teacher did"

"If my sixth grade teacher had explained this material this way, I would have learned this material"

Although, this type of student is experiencing difficulty in mathematics, he is willing to try and feels that if proper methods, materials, and proper type of examples are given, he believes that he can learn mathematics. They are constantly looking for somebody to break the cycle of failure in mathematics. These people are always eager to talk to math teachers who are willing to listen. They are not particularly afraid of math or math teachers, but they do not want to repeat the same experience of failure. They really need help and a good math teacher can usually help them. In that sense, they are easier to teach.

b. Global Math Anxiety

An individual who is experiencing failure in mathematics and may also be doing poorly in other academic areas, but he admits to having anxiety about mathematics only. Generally, this person is an anxious person to start with (may have an over-all anxious personality), but this anxiety is manifested in the area of mathematics. This is so because in mathematics this person is forced to give exact answers and that brings out the underlying anxiety. The person may begin to demonstrate anxiety in other areas also if this person is asked to give exact answers such as: exact spelling, exact dates in history, places in geography; strict chronological details of events. Since that is not the case in other subjects, the person does not develop anxiety about these subjects. Only in the area of mathematics, from the early school experience, a person is forced to give exact answers, and if the person is not able to do so he thinks that he has difficulty only in the area of mathematics and he begins to manifest anxiety in the area of mathematics.

People of this type of anxiety have difficulty in overall mathematics, therefore, their response to mathematics is much more global in nature and much more intense. This kind of anxiety we call **global mathematics anxiety.** These persons are willing to accept this even without solicitation. For example, these persons when referred to the Center for evaluation of their mathematics learning problems, even before we begin the evaluation, make comments about their incompetence in mathematics and their intense feelings about it. Persons with global math anxiety are apt to say:

"I can't do math"
"I just don't understand math"
"Math makes me sick"
"Complete blackout"
"So and so is superbright, he is really good in math"
"You would not believe it, math makes me throw up"
"I will always stay away from math. I hate math."

Global math anxiety persons are less willing to deal with the specific math deficiencies, because they are not aware of what they are. They just do not like mathematics. Period. They will spend great deal of time and energy and make decisions to avoid math.

Unlike specific mathematics anxiety successful experiences can reduce it, this type of math anxiety can not be taken care of by providing just few limited successful experiences in mathematics. At this level one has to provide counseling to change the attitudes of the person who is suffering with math anxiety. The problem is compounded by their negative attitudes towards mathematics learning in particular and learning in general.

Many authors have observed that

i. That the mathematical aspect of math anxiety (specific math anxiety) is much more dominant than the psychological aspect (global math anxiety) of the problem;

ii. That the long run nitty gritty math is the source of the problem and a sympathetic mathematics teacher is needed to remove the clouds.

iii. The math anxiety should be of concern to mathematicians since they are the ones who can do more to help math anxious students than anyone else.

iv. Specific mathematics anxiety is easier to deal with. When one provides successful experiences in mathematics at some level (even at a lower level than chron-

> ologically expected mathematics content) to this type of student, he/she may begin to also the anxiety and begins to feel better about mathematics and him/herself.

Sometimes, for this reason, in our work with math anxious individuals, we start with simple algebraic concepts before the student is asked to master all the arithmetic concepts. Beginning with arithmetic concepts may trigger their anxiety. Alternately we take simple arithmetic concepts and show them the concepts with the help of effective concrete materials. As a result of these strategies, we observed almost immediate feeling of belief that they are capable of learning mathematics. We build on this newly acquired confidence and by taking appropriate digressions to make-up for the arithmetic deficiencies. These successful mathematics experiences result in further building of confidence and reduction in mathematics anxiety. Here is an example of a successful experience illustrating this concept.

> "For my first year at Millfield, I was in a very low mathematics class and I was not studying physics. One of my maths teachers seemed to appreciate that I was bored with maths, not because I could not do it but because it was not stretching me enough, so I was moved up to one of the higher maths classes taught by the head of the department, Mr. Sherlock. He was an extremely good teacher and soon I became much more interested in maths than I was in either biology or chemistry" (Jansons 1988, p. 500).

2. The Role of the Math Teacher

As the above example clearly shows, a good math teacher is almost always the key for solving the math anxious student's problem. The math teacher should try:

a. To identify the type of math anxiety.
b. To identify the cause and source of the math anxiety.
c. To figure out the strengths and weaknesses of the math anxious person.

Good teaching is the key to the alleviation of any kind of math anxiety but much more successfully on the specific type of math anxiety. For, the student generally has appropriate cognitive skills. It is also true that math teachers are usually unable to help a person experiencing global math anxiety. First, because the teacher may not be trained to handle emotional problems. Secondly, time or resources may not be available to attack socio-cultural problems on individual basis. Therefore, this type of math anxiety students should get help from teachers and counselors, both. Successful mathematics experiences are not enough. Because of global math anxiety, the success on one concept may not transfer to another concept. When teachers do not recognize this fact they may even create the resultant mathematics anxiety. For example,

> "So, at the age of 11, I went to a local secondary school, where, at least to begin with I was able to study biology, chemistry, and physics all as individual subjects. However, I was soon prevented from doing physics by a teacher who thought I knew nothing, mainly due to my attempts to point out when he was wrong. If he had only consulted his reference books he may have had quite a different opinion of my ability" (p. 499).
>
> "The grammar school I failed to get a place at had merged with a neighbouring secondary modern school to become a new comprehensive school, and eventually my mother succeeded in getting me transferred there where we believed the teachers would be better. Although this was certainly true, many were reluctant to believe that I was much more intelligent than previous school had given me credit for. For a while I had remedial English lessons with a psychologist at Peterborough Hospital, who believed it to be important for me to accept that I did have a low intelligence, although she claimed this did not mean I could not have a fulfilled life. Not surprisingly these English lessons were not a great success.
>
> "Eventually my mother and I made contact with the first person who really seemed to understand my difficulties: Prof. Oliver Zangwill of the Department of Psychology at Cambridge University. He seemed to have no difficulty at all in seeing that I was very intelligent. His backing helped to convince some people that my problem ought to be taken seriously, and at 14 I started yet another school, namely Millfield in Somerset. This school had an extremely good remedial English department run by Mr. Lynn Lewis. . . .However, there were clearly some very important differences between his [Lewis] attitude and that of my earlier teachers. First, he [Lewis] believed that I was intelligent and that it was not ridiculous for me to want to go to university. Secondly, he never promised a complete cure; he set realistic targets for me and made it clear that I would always have to work much harder than most people. In short, he was very believable" (Jansons 1988, p. 499-500).

The impact of good teachers is felt throughout a student's learning career from grade school to the university education. For example, when Jansons was admitted to his Ph.D program at Cambridge University after completing his undergraduate work there, he felt Dr. John Rallison was responsible for his success in completing his Ph.D program. As he writes

> "...I started a Ph.D. in fluid mechanics under the supervision of Dr. John Rallison.
>
> "Rallison was an exceptionally good supervisor in many respects: he allowed me as much freedom as I wanted while doing my Ph.D. and he had a very healthy attitude towards my reading and writing difficulties, as well as being talented mathematician himself. Many people around did not think I would be able to complete a Ph.D. because of these difficulties, but, like me, Rallison believed in dealing with problems when they arose and not wasting time and energy attempting to see too far into future. My research went very well and after the first year I had finished the piece of work that had been planned for my Ph.D." (Jansons 1988, p. 501).

3. The Role of the Counselor

Global mathematics anxiety is much more difficult to deal with. It is not enough to provide successful mathematics exprienes but it is also important to deal with the underlying reasons for it. Which in this case is either the anxious personality or the lowered self-esteem because of failures in mathematics that are caused by the socio-psychological reasons, not just because of mathematics difficulty. Intervening with these kinds of math anxious people can be thought of as an attempt to reduce the anxiety to the functional point. Psychological interventions are necessary to reduce the global fear of math to specific, identifiable deficiencies in mathematics, which then can be remedied. It is easier to provide successful experineces when specific deficiency in mathematics have been identified.

In the global mathematics anxiety situation the complaints about and reactions to mathematics are so global, intense, and nebulous that one can not really attend to the problem of remediating mathematics problems. The resultant failure in mathematics has affected the person's self-esteem. Therefore, if the self-esteem is not improved the successful experiences in mathematics do not result in lowering of mathematics anxiety. For example, a tutor may help the student to do well on algebra but even after admitting that he is doing well on algebra, the student will say "he is going to do poorly in geometry." Then the tutor helps him in geometry, but then the student will say "but, I am going to fail in calculus." This situation will continue. The anxiety about mathematics continues to affect student's attitude towards mathematics and the fear is being shifted form one mathematical area to another. In such situations, successful experiences in mathematics must be combined with some type of counseling to help the student in developing self-esteem and dealing with the underlying anxious personality. Here team effort: a good math tutor and an effective counselor (with some understanding of math learning problems) is the answer to the global mathematics anxiety. Counselors can provide a bridge the mathematics tutor needs to help remediate the mathematics problem.

As good teachers convey the excitement of the subject, some teachers who are uncomfortable teaching arithmetic, will convey their insecurities to the student and therefore, may create anxiety about the subject. Still other children identify with a parent who has had difficulty in arithmetic, and then they begin to feel that it is alright to do poorly in mathematics and begin to dislike it. Many a times parents repeat the stories of their failures (sometimes even with pride) in mathematics to their children showing that it is alright to fail in mathematics. Some parents consider it fashionable and "a badge of honor" to admit and declare their incompetence in mathematics. It sends a wrong message to children.

4. Self-esteem and Mathematics Learning

Since, mathematics anxiety is a symptom during the early school years, it is easier to deal with it then. Early identification of children for their difficulties in mathematics (earlier than when math anxiety begins to be a causative factor) is a better way of dealing with mathematics learning problems and avoiding the ensuing effects of the resultant math anxiety. Well planned and appropriate early intervention remedial and instructional programs for children having problems in learning mathematics result in successful learning experiences. This success improves a child's self-esteem. Which in turn reduces a child's math anxiety.

In addition to the development of fundamental skills, mathematics teachers should attempt to foster, in their students, positive atttitudes toward mathematics. The development of positive attitudes is another major goal of our mathematics program at the Center. Educational literature has revealed that the development of positive attitudes toward the subject being studied is one of today's most prevalent educational goals. Research has also proven that students having positive attitudes toward mathematics will achieve greater success than student having negative attitudes.

5. Motivation

Both concrete and verbal incentives enhance student's motivation for learning and creative thinking, former more effectively than the latter. A child, with math anxiety, needs greater incentives to gain motivation to engage in mathematics learning activities. The best intrinsic motivation is the taste of success. When a child is provided with successful experiences at some level of mathematics complexity, initially even at a very low level, he begins to gain confidence. This self-confidence works as an important intrinsic motivator. Children produce the greatest number of positive responses when rewarded and encouraged for their performance and far fewer ideas when they received no reward for their effort. Research findings add creative thinking and reduction of anxiety to the growing list of behaviors that are affected by the taste of success.

Individual attention has consistently been shown to raise a student's confidence and self-esteem, the building blocks for better academic performance. Enhanced academic performance expands options and choices, generating the dream that help young people set goals. Students don't drop out when they have goals and have tasted success.

III. Factors Responsible for Mathematics Learning

The sources of children's learning disabilities are far from clear. It is presumed, as we saw in the previous sections, that learning disabilities, particularly dyslexia and dyscalculia, stem from dysfunction of the central nervous system, but this does not provide sufficient explanation for the specific problems that many children and adults face in learning and using mathematics in general and arithmetic in particular in spite of normal neurological and neuro-psychological results on examination. As there is no single set of symptoms that characterize all dyslexic children or one single type of brain failure that causes these symptoms, similarly there is no one cause of the dyscalculia and other related mathematical difficulties manifested by children.

There is a whole complex of reasons and causes for problems in mathematics learning.

A. **Cognitive Development and Mathematics Learning**

The learning disabled (LD) youngster is often described as having difficulty applying the appropriate strategy to learn. He may not, for example, have a systematic strategy for solving a problem. Or he may not be able to perform mental computations because he does not use appropriate cognitive strategies (for example, verbal rehearsal strategies). There is a long list of cognitive strategies and concepts that play a role in learning mathematics. The learning disabled student is either deficient in the strategy itself or in the ability to produce that strategy in the proper situations (Flavell, 1970).

Let us take a few examples from the type of mathematics problems that we see routinely. When 6 to 9 year old students are given the common missing addend problem $(2 + \underline{\quad\quad} = 8)$ they give the following answers:

(i) $2 + \underline{10} = 8$, (ii) $2 + \underline{1} = 8$,

or (iii) $2 + \underline{6} = 6$.

Each answer is incorrect but these incorrect answers indicate the level of these children's thinking. Each one of these reflects and gives some clues to the reasons for that child's difficulty and the incorrect answers.

The *first* answer indicates that the child's thinking is *centered*, i.e., he is focusing on one concept at a time rather than considering all the concepts involved in the problem. In this case the child centered his attention on the operation of addition and therefore he added the two numbers in the problem, irrespective of the position of the numbers in the problem. The condition of centeredness is an inhibiting factor in learning *secondary* concepts in mathematics. [The concepts in mathematics are of two types: *primary* and *secondary*. Primary concepts are those concepts that are independent of each other and are autonomous in nature. Addition, subtraction, multiplication, and division of single digit numbers and counting are examples of such concepts. Secondary concepts on the other hand are combinations of several primary concepts. Addition of 2 2-digit numbers with regrouping is an example of a secondary concept as it involves addition and regrouping.] Centered children have difficulty in dealing with secondary concepts. And since, except for the very early elementary concepts, most mathematics concepts are secondary, these students continue to have difficulty in these concepts till they lose their centeredness due to their cognitive growth. But the centeredness and the resultant mathematics failures in these years have inhibited and even brought down child's preparation for mathematics learning. If no special teaching strategies have been provided for these students they develop high levels of mathematics anxiety. Parents and teachers continue to have quite high expectations in spite of their difficulty in learning mathematics thereby compounding the problem and adding to the resultant mathematics anxiety.

The presence of centered condition, beyond 7 years of age, is due to lag in child's cognitive development. The above examples illustrate that the first factor that affects a child's learning ability and therefore his capacity for learning mathematics is dependent on the student's cognitive development. Whenever there is a mismatch between the level of the cognitive complexity of the concept being taught to the student and his cognitive development one sees difficulty in mathematical conceptualization and therefore, achievement. A higher level of cortical functioning (higher level of **cognition**) is necessary for the appropriate cognitive development and therefore mathematics achievement. Cognition is an all-encompassing term used as a concept heading for such activities as reasoning, problem solving, and thinking. It also includes multitudes of cognitive actions (e.g., discrimination, analogy, transitivity) that are employed when one acts cognitively as in learning, doing and applying mathematics.

The learning disabled seem to be less efficient in the performance of cognitive tasks and as a result lack the capabilities to solve mathematics problems successfully. They tend not to use the cognitive abilities they possess. One needs to see whether the lowered performance is due to lowered cognition of the student or lack of efficiency in their use of cognition.

The **first** consideration in teaching and learning of mathematics, therefore, is that the remedial teacher/tutor identifies thie child's **level of cognitive development** and considers this information in planning a remedial instructional program.

B. **Mathematics Learning Personality**

Each one of us processes information, and therefore mathematics information, differently and uniquely. This processing difference defines the unique way we learn mathematics. Many individuals are good in algebra in secondary school and others are more proficient in geometry and some people find both algebra and geometry equally difficult or easy. A small number of students in each class are good in both. One finds that a large number of students do quite well up to about the age of twelve but begin to have difficulty in secondary school mathematics. What is the reason for these differences in student's mathematical performance? Why does the same person find one aspect of mathematics difficult and the other easier? Similarly, why in the case of persons of the same level of intellectual (cognitive) ability, does one find different patterns of development and achievement in mathematics.

The reason for this differential achievement in mathematics seems to lie, at least partially, in different ways of processing mathematics information.

Students who are sequentially oriented, i.e., those who focus on procedures or deductive reasoning, process information from components to whole, they do better in arithmetic, some quantitative aspects of algebra, and other similar types of mathematics concepts. We identify this type of student as having a *quantitative mathematical learning personality* (Sharma, 1979). This student is generally very good in language and verbal expressions; he is good in solving problems bit by bit (in components); his approach, typically, is to decompose the given problem into pieces, solve them and then recompose the component solutions to generate the solution to the given problem. He is good at quantifying the given information in problems and then solving those problems by applying mostly sequential procedures, quantitative operations and approaches. He is especially good at operations that build up sequentially.

On the other hand, students who focus on the visual/spatial, wholistic, and inductive aspects of mathematical information they, generally, do better on geometrical concepts and the application aspects of mathematics. These students, generally, may have difficulty in much of the elementary mathematics if the instructional approach is heavily sequential. We identify this approach to mathematical information processing as *qualitative mathematical learning personality*. This type of student, if taught using inductive, visual/spatial, and pattern recognition methods, learns mathematics better. This student uses intuitive approaches in solving problems. He uses spatial and perceptual cues to generate information about the problem. As a result, he looks for general patterns in the given information in a problem. This student learns most of mathematics by seeing interrelationships between concepts and procedures. But this student faces difficulty in following sequential procedures (algorithms) and makes mistakes in details of procedures. He also makes on the average, compared to the quantitatively oriented student, more careless errors. Many a time, he does not show details of his work, invents and uses shortcuts by eliminating steps and deriving several steps mentally. This student naturally finds geometrical concepts easier to learn and work with, or those concepts in which patterns are evident or possible. He is prone to commit errors by leaving out signs of numbers and manifests somewhat of a sloppy work habit in procedures. He may find procedural work boring and tedious. He may lose interest easily in long-drawn procedures. He generally does not practice enough to achieve an automatic level of mastery. Very often, rather than automatization of a particular procedure and a fact, he derives it when he needs it, as he generally does not practice exercises to reach the level of automatization necessary for fluent performance. In the extreme manifestation of this level of learning style, the student appears to show lack of interest in mastery of a concept. However, he is able to see conceptual connections faster than the quantitative mathematics learning personality student.

To learn mathematics, therefore, involves both approaches: quantitative and qualitative. To acquire a high level of fluency in mathematics work, the elements of both types of learning styles must be integrated, although in a particular individual, one learning style is more dominant than the other.

Every mathematics problem, potentially, can be dealt with by both approaches to teaching: quantitatively and qualitatively, i.e., using standard deductive, algebraic, sequential, procedural recipes or using visual, spatial, inductive, pattern recognition strategies. The following account of teaching a concept illustrates the point.

Jansons writes, in his autobiographical note: *A personal view of dyslexia,*

> The first examples class I was asked to give to the undergraduates at University College involved working out the total resistance of a network of resisters. It turned out that the students were intended to translate the problem into algebra, after spotting the obvious symmetries of the network, and to solve the problem algebraically. However, I manipulated the resister network by mentally cutting, folding, and reconnecting it in a way that would clearly not change the overall resistance, thus reducing a complicated network to a simple one which could be solved immediately to give a numerical answer. At each stage, I drew a diagram on the black-board for the benefit of the students to explain the method. The process of reducing the complicated network to the simple one was completely non-verbal, though just as precise as the algebra for which it was a substitute.
>
> The reaction of the students divided them into two groups. Two-thirds of them were very impressed with my method as they could see that not only was the argument shorter but they also felt it to be more elegant and natural than their own. The remaining one-third were only impressed with the speed with which I solved the problem but did not feel they would be able to generalize the argument to solve similar problems. Maybe the division was essentially between those that found some visualization easy and those that did not" (Jansons 1988, p. 503).

Visualization is one of the key elements of thinking for the qualitatively oriented students (Sharma, 1989).

> From an early age I found that many things were easier to think about without language. This usually, but not always, meant thinking in terms of pictures and was particularly true when trying to make or understand intricate mechanisms. It is also the way my father thinks about things. I soon found it reasonably easy to create images in my mind and to manipulate them and, although I do not have a photographic memory, I developed a very good visual memory. For example, in building Meccano models (an engineering based construction toy) I would try out many possible arrangements in my mind before any assembling of the pieces.
>
> To me abstract pictures and diagrams feel more important than words. For example, in maths I never feel that I fully understand something unless I have found a way of visualizing the system, although this is not usually a substitute for mathematical symbolism. Many of my original mathematical ideas begin with some form of visualization and, once that seems to be a sound model of the system I am analysing, the analysis usually falls into place. Occasionally this fails to happen and then I return to and modify the visual model (Jansons 1988, p. 502).

The identification of a student's **mathematical learning personality** is another important consideration in designing a student's remedial mathematics program. We have observed that matching the teaching and learning styles results in a better use of the remedial instruction by the student and in such a situation optimal results are possible.

C. Mathematics Language

Mathematics is a form of language involving the communication of concepts through symbols. In most cases, the mathematics language is the second language a child encounters after his native language. Among the concepts embodied in this unique language are, at primary level, classification and ordering of objects, quantity representing actions and collections, size, order, relationships, space, form, distance, and time. On a higher level the language of mathematics is used to help in conveying the results of logical thinking and reasoning. It is used in the collection, classification and organization of data, and then in the analysis and interpretation of that information.

Both, the child's native and the mathematics language play an important role both in the conceptualization of mathematical ideas and in the use of mathematical information. When a student complains that he has difficulty in doing mathematical word problems, the problem is not just mathematical. To be proficient in mathematical problem solving, particularly word problem solving, one needs competence in mathematical vocabulary, in understanding the syntax and voice used in the mathematical language, and the ability to translate from mathematical language to the child's native language and vice-a-versa. A student may have a good grasp of the arithmetic concepts but if he has difficulty in translating the English expressions of the word problem into mathematics formulae and equations, he may have difficulty in solving the word problems. Similarly, if a student has difficulty in translating mathematics equations into English language, he may have difficulty seeing the meaning of mathematics formulations of real life problems. Both type of translations, from English to mathematics and from mathematics to English, are essential for proficiency in mathematics problem solving.

Almost every mathematics idea, except simple arithmetic facts, consists of three components: *linguistic, conceptual*, and *skill procedural*. Different and specific approaches of teaching are needed for a child to learn these three components effectively. In the initial stage of learning a mathematical concept the emphasis should be on the conceptual development and language. The learning is interactive; conceptual models help the acquisition of language and the language of mathematics plays an important role in the conceptualization of mathematics. Using quantitative terms in the day to day language interaction prepares the child for the eventual development of the language of formal mathematics. However, mathematics language for some students is easier than other languages and because of its compact nature, some students may excel in mathematics, in spite of their difficulties in reading and writing. For example,

> The very compact notation [of mathematics], which I had no trouble in reading or writing, excited me and allowed me to play around with mathematical ideas for a long time after reading only a very small amount. Soon I became one of the best mathematics students in the school as it was such an easy subject to learn. In some cases the best mathematical ideas of a century could be condensed into a chapter. A little after starting to study maths further than arithmetic I was able to start physics, which was taught by a very enthusiastic teacher called Dr Hanna (Jansons 1988, p. 500).

Therefore, mathematics thinking, to a great extent, is dependent on **mathematics language.** As a matter of fact, mathematics is a bonafide second language, and if we want to help the child to think mathematically we need to help him in the acquisition of the mathematical language: its vocabulary, its syntax, and the translation ability from one language to the other.

D. Levels of Knowing Mathematics

Every mathematical skill is a developmental concept. To master a concept at its fullest, a child needs to understand it at all its levels of manifestations and at all of its levels of difficulty. The mastery of a given mathematical concept passes from intuitive level of understanding to the level where the child can explain how he has arrived at a particular result and can explain the intricacies of the concept. However, to reach this highest level of understanding and mastery, the child goes through several intermediate and partial levels of mastery.

There are six levels of the mastery of a mathematical concept: *intuitive, concrete, representation* (pictorial), *abstract, applications*, and *communication*. At each of these intermediate levels, the mastery is incomplete and regression is always a possibility. Only when a child has reached the final level of mastery — the communication level, can one assume that the student has **really** mastered that skill/concept. Ideally, each mathematics concept should be introduced beginning at the intuitive level and ending at the communication level.

In many of the regular classroom teaching situations, the teacher, instead of introducing the concept at the concrete level, may begin at the abstract form of the concept. And, as a result, the student may face difficulty in learning the concept of the procedure being taught. Even if the child has understood the procedure for solving that problem he may soon forget it. Later, when the teacher begins a new concept he may assume, incorrectly, that the mastery in the previous concepts is still present and therefore he may begin the new concept at a higher level, i.e., at the abstract level, creating difficulty for the student. This cycle continues and eventually the student begins to lose the teacher's explanations. He begins to have difficulty in learning mathematics. Which then results in failure and that develops a fear for mathematics. As a result of these failures, then, he may be identified as a learning disabled student in mathematics. The student himself begins to feel that he has a learning disability in mathematics with a resultant lowering of his self-esteem.

Almost all mathematics teaching activities, in most classrooms, take place at the abstract level. That is where most text books are; that is where most of the tests and examinations are. Because of the abstract nature of teaching many students may regress in the material they have just learned, in spite of the fact that the children have understood the procedures in the classroom. It is easy to forget procedures that are not rooted in understanding of the underlying concepts. Children regress more in the mastery of mathematics concepts and procedures because the teacher did not plant the mastery at all levels of knowing a mathematical concept. As a result, many children do not do well on cumulative examinations and tests. To help our students (our children) learn well and also to do well on examinations and tests, we should,

(1) begin the introduction to the concepts at the intuitive level and take them through all the levels up to the communications level;

(2) make sure that the children have understand the three components of the mathematical concepts: linguistic, conceptual, and procedural;

(3) over-teach every concept. Over-teaching involves: (a) repetition (in normal conditions, for any concept to transfer from short term memory to long term memory it has to be repeated many times, in some cases as much as 18 to 25 times by hearing it, saying it, or using it in one form or the other) and (b) taking the student to a higher level of mastery than required on the tests and examinations. Since the tests and examinations require a child to have the mastery of concepts at the abstract level of understanding, we should take the student to at least application level of mastery. It is better still if we take him to the communications level of mastery.

Presenting a mathematics concept using concrete objects and taking it through all of its intermediate levels to the level of communication is another consideration when designing instructional and remedial programs in mathematics for students having difficulties in learning mathematics.

E. **Prerequisite and Support Skills in Mathematics**

> Quite a good number of dyslexic children make mistakes over arithmetic simply because they lack a secure sense of left and right. For example, confronted with the figure 13, they may be unsure whether the 3 or the 1 should be considered first and therefore may wonder if the number should be thought of as 'thirteen' or as 'thirty-one'; or again, if required to do long multiplication, they may show hesitation as to whether to work from left to right or from right to left (Miles & Miles 1975, p. 51).

The above example shows that following the procedure of the algorithm and number conceptualization can be dependent on the child's ability to identify the left-right orientation (a sub-skill of spatial orientation and space organization).

Everyone who has tried to help a child in teaching him reading knows that one has to help the child acquire readiness for reading skills before actual meaningful reading can take place. This holds true for mathematics also, with the exception that what is a prerequisite skill for subtraction becomes a support skill for the concept and algorithm related to division and fractions. As in reading, there are readiness skills to facilitate the reading skills, so there are certain non-mathematical skills that are necessary for mathematics conceptualization. When these skills are present in the child's repertoire of skills, mathematics learning becomes easier. And in the absence of these skills, mathematics learning is transitory. One day a child may recall the concept and the procedure and a few days later he may not be able to do so. In that sense these skills are "anchoring" skills; mathematics conepts get anchored to these skills.

Of the many reasons that could contribute to a child's learning difficulties in mathematics the ones which can be dealt with most easily and with little special training are these pre-requisite and support skills. Before a child can meaningfully understand and use a mathematical concept he needs these tools which are generally not provided for in a standard school mathematics textbook or even in an ordinary classroom. Therefore, in any remedial/instructional program, whether the child is eight years old or fifteen years old, one needs to devote a portion of the remedial/instruction time (whether it is the parent working with the child or a special education teacher/tutor) in helping the child acquire these skills. Some of the important prerequisite and support skills are:

- following sequential directions
- classification and class inclusion
- ordering and sequencing
- spatial orientation/space organization
- estimation
- visual clustering
- pattern recognition and its extension
- visualization
- deductive thinking
- inductive thinking

Let us look at an example of long multiplication from the perspective of demonstrating the importance of prerequisite skills. Let us find the product of 96 x 47. To find this product, the student has to follow more than sixteen sequential directions. In this case, a student may get the incorrect answer either because of his difficulty with left and right orientation (a sub-skill of spatial orientation/space organization) or because of his difficulty in following sequential directions in the multiplication algorithm or not remembering the needed facts. Therefore, children's incorrect answers in mathematics problems expose the existence of their mathematical difficulty but they do not tell us neither the nature of the problem nor the causes of the problem. As a result, although the children's incorrect answers may be similar, the approach to remediation may not be the same.

Two children who manifest the same problem may do so for two different reasons. If a student has difficulty in following sequential directions, then he may have difficulty in most of arithmetic algorithms and later in the procedural parts of algebra and in the development of geometrical proofs.

Almost all of school mathematics is dependent on a student's ability to follow sequential directions. One obvious way to help children having arithmetic difficulties is to help them develop their sequencing ability and improve their sense of left-right awareness.

> For most dyslexic children anything which gives them a firmer sense of left and right is likely to prevent errors in arithmetic (Miles & Miles 1975, p. 53).

The same thing applies to most of the other prerequisite skills outlined above. Learning mathematics in the absence of these prerequisite skills is like having toy balloons, if they are not tied to something they will fly away. When these prerequisite skills are present, the mathematics concepts get tied to these skills and then the mathematics concepts do not fly away.

Developing *prerequisite* and *support skills* before or during mathematical instruction is another important consideration in teaching children having learning problems in mathematics.

IV. Remedial/instructional Approaches

A. Helping a Child With Mathematics Learning Problems

Mathematics is relatively hard for some pupils. But for future success in demanding professional fields and careers a good foundation in mathematics is very important.

We believe, along with language, the ability to deal with quantity and numbers not specific mathematics knowledge is a built-in function of the human brain. Numbersense is inherent in human beings. Arithmetic and mathematics is acquired through the use of this ability. Not every child can become a mathematician but every normal child who can read meaningfully can acquire mastery in basic arithmetic: addition, subtraction, etc. It has been found that all children love counting — at least until they start school. That is the beginning of number operations.

For a child learning arithmetic and mathematics can be difficult and frustrating, but so can learning to tie a shoelace. How did a parent help the child learn that? Patience and persistence was the answer then and that may still be required. A parent should not let the children leave home with their numbersense shoelaces untied.

Many children and adults who say, "I can't do math" or "I hate math" really can do mathematics Yet they have become afraid as a result of many unsuccessful experiences and also because they were not taught how to learn mathematics effectively.

What can one do if child is in trouble in a mathematics class? The best choices are supervised study or tutoring for the child.

As outlined earlier, a child may have difficulty in learning mathematics for several reasons: lack in cognitive ability; mismatch in the mathematics learning personality of the student with the teacher's teaching style; absence of prerequisite and support skills; lack of development of mathematics language; and lack of mastery in basic arithmetic skills. Therefore, before any tutoring begins one should identify the causes of the mathematics learning problem. Without meaningful diagnosis the remedial effort may be a wasted effort at worst and getting minimum result at best.

Learning mathematics effectively means acquiring particular skills. That include: listening as a mathematics skill; prerequisite skills; problem formulation; understanding the language of mathematics; learning from homework; decoding formulas; estimation; preparing for and taking a math test. Mathematics learning requires specialized methods of learning and study habits. A student with difficulty in mathematics needs to learn these skills. For example, such a student must start the homework as soon as class is over so that he still remembers what the teacher has shown him. If this is not possible it should be done no later than that same evening.

Whatever the choice for providing help for the child, it must be made before the student is in desperate trouble. It must be carried out on a regular schedule until the student feels comfortable in mathematics. Although looking for help just before the tests and examinations is not very beneficial, if the child does need help one should provide it. Parents should encourage the child to tell them about the schedule of examinations in advance. The parent should find out what kind of examinations he needs to take during the year and other special examinations.

Our general experience is that difficulties over calculations can be a nuisance, but that they need not be an insuperable obstacle to the learning of mathematics. Here, as in other areas, it is perfectly possible for a person with dyslexia and/or dyscalculia to achieve success in mathematics if proper diagnosis of the problems is made and efficient teaching methods are used.

As in other learning situations, we can diminish the child's desire to learn mathematics by limiting the experiences to which we expose him. The quality of any thinking is the funciton of the uses one puts into it. When a child is provided a variety of experiences he not only gets the opportunity to learn new things but he also has the setting to test and verify his previous learning and the methods of learning.

B. Characteristics of a Tutor for Mathematics Learning Disabled Students.

When one sees delay in acquiring even the rudiments of numerosity, it becomes essential, then, to provide that child special instructional and remedial strategies. This help must be provided by a person who knows how children learn mathematics, why some children manifest mathematics difficulties and is aware of special teaching techniques to reach these children. "Simply correcting the child's mistakes and relying on constant repetition is unlikely to be effective" (Miles & Miles 1987, p. 1). One should know the relationship of effective ways of dealing with the problem and the child's progress. Our experience has taught us that the effectiveness of remedial strategies used is dependent on the understanding of the nature of the child's problem. Many a times the same strategy may not be effective with two children manifesting the same problem. The tutor should know that the manifest nature of the problem of the two children might be same, but the causes of that problem might be quite different.

The initial teaching and initial remedial work in mathematics can be done by a person with a limited experience and training. What is needed is that tutor has empathy with the child and his problem, feels confident in reaching the child, and knows the mathematics content well. But for successful remediation of the complex mathematics learning problems one needs to know (a) how children learn mathematics, (b) the nature of mathematics learning problems and (c) efficient and appropriate ways of dealing with them.

It is a common assumption on the part of many educators that if a teacher knows mathematics then she will be able to work effectively with children who are having problems in mathematics. Or, if she knows how to deal with the child's reading problems then with little training in math she can deal with his mathematics problems also. These assumptions are accurate only in a limited number of cases. A person might be an expert in mathematics but still may not be able to diagnose or remediate a child's mathematical problems or even design an appropriate mathematics learning program. To be able to work with children having learning problems in mathematics, a person needs to know the way children learn and acquire mathematical concepts, the nature of the mathematics problems and type of difficulty that the children especially encounter in mathematics.

Parents and teachers can help on the morale side by encouraging the child to take his time and not to panic. Some preliminary practice at doing arithmetical calculations under pressure of time may serve to acclimatize him to the examination atmosphere.

Compensatory mechanisms such as counting on fingers, making marks on a paper and other secondary devices that children either invent for themselves are indications of something is a miss, i.e., the child does not have appropriate models for mathematics concepts. One can begin with these models, that is, let the child use these mechanisms, but the parent and the teachers should continue to work in developing appropriate and efficient models which are better organizers of information. Till a student has developed efficient methods of calculations, it is unwise to suggest to the child that such aids are inherently undesirable and insist the child to drop his compensatory strategies. We should point out from time to time that these mechanisms are inefficient and have drawbacks and the parent and the teacher should continue to help the child develop better strategies. It can be explained that in arithmetic, there may be different ways of arriving at the correct answer, and that for a dyslexic child the standard, quick method is not necessarily the right one.

V. **Parental Involvement**

Several studies and research reports during the last decade clearly show that a child's success or failure in school is, to a great extent, contingent upon the learning that takes place at home. Recent research proves the importance of parent involvement for the academic success of children.

> Parents are their children's first and most influential teachers. What parents do to help their children learn is more important to academic success than how well-off the family is" (AMP 1989, p.1).

> "After being at school for a few years, however, I found that, although I certainly seemed to be more intelligent than all the other children there, they were able to learn to read more easily than me. So after school, my mother would give me extra lessons." (Jansons 1988, p. 498).

This does not mean that the school does not play an important part in child's academic success but what it means is that the link between learning and parental involvement in a child's learning process is an important one. The implication is that parents should take an active role in their child's learning. Every parent, irrespective of educational background, training and economic status can participate in their children's learning meaningfully.

> "Parental support and encouragement are vital to children's mathematical development and success in school. Social, economic or educational status of parents does not have as important an effect on their children's learning as what parents **do** with their children" (AMP 1989, p. 4).

Every parent must assume personal responsibility for educating one's own child. Here we do not mean taking one's child out of school. It means that a parent should be constantly aware of what is happening in the child's education. Passing the buck to "educational experts" and other custodians of education of children is not sufficient. We are not saying that parents should be totally responsible for educating the child, what we are suggesting is that every parent needs to monitor the learning, development and achievement of his/her child very closely. Waiting for the nominal reports and parent-teacher conferences to tell one about one's child's progress is taking risks. In many situations waiting for the school's input might turn out to be costly both in emotions and finances. This applies whether the child is in a public or a private school. In today's schools there is no guarantee that a child is being taught by a teacher who has the required training in mathematics teaching. Most classes in most schools do have qualified math teachers (i.e. properly and adequately trained in teaching mathematics) but there are many classrooms where either the teacher is not qualified or just does not have interest in teaching mathematics or is afraid of mathematics. And training does not guarantee good teaching either. One needs to find it early on before one discovers that no or little learning has taken place. A mathematically anxious teacher can actually develop math anxiety in a child instead of fostering in children positive attitudes toward mathematics. Parents need to intervene in such a situation. A parent should not wait for three months for the first formal report about the child's learning progress to come out. The parent is the one who has the ultimate responsibility of one's child's welfare. Parent has to be the guardian of one's child's education in the true sense of the term.

A. Nature of Parental Interaction With Children

Learning is a normal child's play and in turn play is learning for every child. Every child believes in it and shows this through its curiosity to know every thing and anything. For a child, learning is the greatest adventure in life. Learning is desirable, vital, unavoidable, above all, it is child's life's greatest and most exciting game. The child continues to believe this and will always believe this-unless adults persuade him to think otherwise. We, adults, change children's attitude about learning and we do this to children in many ways. Sometimes we do it by saying that learning is going to be difficult. Sometimes we do it by expressing our own distaste for learning; which sends a clear message to the child that "if I don't like learning how can you?" Sometimes we do it by forcing the child to do what we want him to do rather than what he would like to do. As a result, we create conditions where the child loses interest in learning. Other times we do this by giving the child tasks that are beyond his ability which then result in failure and eventually in creating a distaste for learning.

> "International studies of mathematics education have indicated that parental attitudes toward mathematics strongly influence the achievement of children" (AMP 1989, p.1)

If I constantly tell my child that I did not like mathematics and failed in it, I am conveying the feelings of failures and negativity. It does not have to be this way. We can change this. We can change this by changing the nature of our interaction with children. From the type of interaction that conveys the attitude of failure to providing the feelings of self-worth and optimism about learning. I am not saying that one should lie about one's inadequacies in mathematics to children, but one does not have to make a regular production of one's incompetence. One can admit it. For example, one can say that he had difficulty in learning mathematics and admit that mathematics is a difficult subject for him. But then one should add that "we should see if you can get help in it." In other words, the approach has to be: "let us see what we can do so that we can understand this mathematics concept together" or "let us find someone who can help us to learn this better." We should learn something for the child's sake. What is wrong in that? Many parents devote enough time in coaching little leagues and boy scouts and sundry other things. It is not a bad idea to spend some time in learning mathematics also. It is never too late to learn. One should approach all learning positively.

Any subject matter learning can be made a joyous experience if approached properly. And this applies to mathematics also. But it is not the subject matter that really counts; it is persevering, finishing a task, the feeling that you can really do something — those are the things on which success is really built upon. For a child to experience success, we need to convey that these traits are important. In a sense we might have to change ourselves, if we really want our children to experience success in any learning activity.

The Mathematical Association of America and the National Council of Teachers of Mathematics (NCTM) appointed a joint committee in 1986 on Parent Involvement, charged with drafting a statement on the importance of parental involvement in student learning in mathematics. Following are the recommendations from the committee:

Parents must take an active role in their children's education by:

- ★ discussing regularly with children their classroom activities and listening carefully to their explanations of what has been learned;
- ★ providing time and a place for doing homework assignments;
- ★ encouraging their children to persist when the work becomes difficult, but not to expect parents or tutors to do the work for them;
- ★ participating in parent-teacher conferences;
- ★ taking advantage of opportunities to visit mathematics classes;
- ★ engaging with their children in appropriate family games, puzzles, and other activities that use a variety of mathematics skills.

The most important rule in helping one's child in mathematics is that both the parent and the child approach the learning joyously, realizing that what a superb game mathematics can be. It should be done as a reward rather than a punishment; learning is and should be a rewarding experience for both. The child should not get the impression that it is a punishment. It should be a privilege not a denial. The wise parent will always make this clear to the child. The opportunity to play math games should be given only to children who are manifesting good behavior. Children who are behaving badly should be denied the opportunity to play those interesting games. That is why it is helpful to connect the games and puzzles played at home with math taught at school. A parent/teacher can help make the math connection so that the child will find it easier to learn and like math.

The length of time that you engage in the mathematics activity and mathematics games should be short. The parent must stop each session before the child wants to stop. The most important thing a parent can do is to reinforce the math lesson the child receives in school. One does not have to be a teacher or a mathematical genius to know that math is learned naturally by the inventive, curious mind.

The above suggestions are based on the premise that a parent can make a difference in a child's attitude toward mathematics. As a parent, you can help your child in mathematics even if you are not very good at it yourself. There are two vital considerations:

1. Your attitude about mathematics and how to teach it.
2. The type of mathematics instructional materials you use.

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- Mathematics learning style,
- Preparation in pre-requisite and Support Skills for mathematics learning,
- Mathematics language, and
- Arithmetic/mathematical level of functioning and achievement.

One or both parents of the child is (are) requested to be present during the evaluation, (with the child's consent). An oral report is provided to the parents summarizing the findings immediately after the evaluation. A written report follows a week to ten days later. If requested a tutorial program is planned for the child at the Center. The frequency and the length of the tutorial program is determined by the out-come of the evaluation. The fee for evaluation is $275 and for tutorial is $50-$60 per hour or $25-$30 per half-hour. Younger children are scheduled from 1 to 2 half-hour sessions per week and older students 1 to 2 one-hour sessions per week. (Fees are subject to change from time to time).